Singular Trajectories in the Two Pursuer One Evader Differential Game

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***Abstract*—The Two Cutters and Fugitive Ship game posed by Isaacs is revisited again. We discuss and analyze the singular configuration of this two-pursuer one-evader differential game. This paper addresses the question of whether or not either player has the ability to exploit the dispersal surface. Specifically, we investigate the case where the Evader effectively stands still (e.g., by dithering in a small neighborhood). We show that the canonical optimal pursuit policy yields chattering in the discrete-time version of the game. As the timestep approaches zero, the capture time approaches the Value of the game, and thus the Evader is not penalized for standing still. Implications on related scenarios are discussed.**

I. Introduction

> "My *dictum* is that the emphasis for two-player differential games with perfect information should be on singular surfaces. Through them will the theory be completed."
>
> — Rufus Isaacs, 1969 [1], [2]

The singularities that can (and often do) occur in the Value function is one of the major distinguishing factors of differential games in the broader context of optimal control. At least once, Isaacs, in his book [3], goes so far as to say that the theory of differential games generalizes optimal control, of which he refers to as "single-player games". By Value function, we mean the minimax value of the cost/payoff of the (zero-sum) game.

In this paper, we dive deeper into one such singularity which appears in the Two Cutters and Fugitive Ship game (2CFSG) posed by Isaacs [3] (and also by Steinhaus [4]). This is a two-player, zero-sum pursuit-evasion game with two faster Pursuers and a single Evader, all players having simple motion. The Pursuers cooperate fully, that is, their individual control actions jointly comprise the strategy of a single player. Capture of the Evader in minimum time is sought by the Pursuers, while the Evader seeks to "live" as long as possible. Since the Pursuers are faster, capture is inevitable. As will be discussed further, the singularity arises when the Evader is positioned anywhere along the line joining the Pursuers. In this configuration, the optimal choice in heading for all of the agents is not unique, which is a symptom of the singularity. Thus, the Evader, for example, has a choice between two different headings, both of which will ultimately lead to the same capture time under optimal play. This is characteristic of the dispersal surface (DS), one of the singular surfaces which Isaacs discussed [2], [3].

In his treatment of the Wall Pursuit Game, wherein a single Pursuer captures an Evader constrained to run along a flat wall, Isaacs discussed what he referred to as a "perpetuated dilemma" [3]. The dilemma occurs when the Pursuer and Evader lie on a line that is perpendicular with the wall. Like the 2CFSG, this is a situation in which the game-optimal choice of heading is not unique. This dilemma is perpetuated when both players choose the same direction (e.g., both players choose up) because they will still be in the singular configuration after some amount of time holding course. The symmetry is, of course, broken after a single mismatch in the players' controls, after which optimal play resumes until termination of the game (i.e., capture). If we conceive of the continuous game being implemented in discrete time (with piece-wise constant controls/headings), then it is clear that the probability of both players choosing the same control over the entire playout of the game approaches zero very quickly as the timestep shrinks. Thus, the perpetuated dilemma is of little practical consequence, since a single wrong guess by the Pursuer (e.g., Pursuer aims up, but Evader chooses down) incurs only a small loss in performance for the Pursuer, which shrinks as the

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This paper is based on work performed at the Air Force Research Laboratory (AFRL) *Control Science Center of Excellence*. Distribution Unlimited. 25 Jan 2019. Case #88ABW-2019-0359.

timestep shrinks.

A perpetuated dilemma occurs in essentially the exact same way in the 2CFSG (i.e., consider the second Pursuer to be a reflection of the first along a virtual wall). We seek to address what happens when the Evader deliberately forces the game to stay on the DS, thereby perpetuating the Pursuers' dilemma of choosing which direction to head and inflicting small, but continual, losses to the Pursuers.

Isaacs' geometric method of solving the 2CFSG (for the case of simultaneous capture by both Pursuers) yields the canonical strategy of all three agents heading to the further of the two Apollonius circle intersections [3]. The 2CFSG was revisited by Garcia *et al.* in [5] wherein a candidate Value function was derived and shown to satisfy the Hamilton-Jacobi-Isaacs equation and is continuous and continuously differentiable everywhere (except for the DS). Fuchs *et al.* analyzed a slight variant of the 2CFSG in which the Pursuers are endowed with a small capture radius [6]. Note, when we refer to the 2CFSG, the termination/criteria requires one or more Pursuers to be coincident with the Evader (i.e., "point capture"). Most recently, the 2CFSG was treated in [7] wherein surfaces separating the state space into regions of solo capture by one Pursuer or simultaneous capture by both are derived. We refer to the separation of these different end-game scenarios as the solution to the Game of Kind. Both [5] and [7] treat the Game of Kind as well as the Game of Degree (i.e., determining the minimax capture time), but the latter utilized a reduced dimension state space which will be employed here. Part of the motivation for delving into the intricacies of the DS in the 2CFSG is the fact that the possibility for non-uniqueness of the optimal strategies appears to increase with additional agents. A primary example is the multiple-pursuer single-evader scenario considered in [8], [9]. There, the Evader appears to have an additional choice under some circumstances: staying put. Because the so-called *regular solutions* are undefined along singular surfaces, we seek to understand whether one or other player has some inherent advantage in its ability to exploit the singularity.

The paper is organized as follows. Section II gives the formulation of the 2CFSG, defines the singular configuration, and gives the regular solution of the 2CFSG. Section III then analyzes the situation in which the Evader chatters around some small neighborhood; as the neighborhood shrinks, the Evader stands still. Section IV provides some numerical simulations and discusses results. Section V concludes the paper with some remarks on these results and their applicability to other scenarios.

II. Problem Formulation

In general position, the state of the system is comprised of the Pursuer-Pursuer half-distance, x_P, and the relative position of the Evader in the rotating reference frame, x_E and y_E. Figure 1 shows the reference frame, system states, and control variables (headings). The x-axis is affixed to the instantaneous positions of the Pursuers, and the y-axis bisects them. By convention, the y-axis is directed such that $y_E \geq 0$; we use E to denote the evader's position. Similarly, we have $P_1 = (x_P, 0)$ and $P_2 = (-x_P, 0)$, by convention. We assume that the Pursuers share the same speed; the Evader/Pursuer speed ratio is denoted by μ.

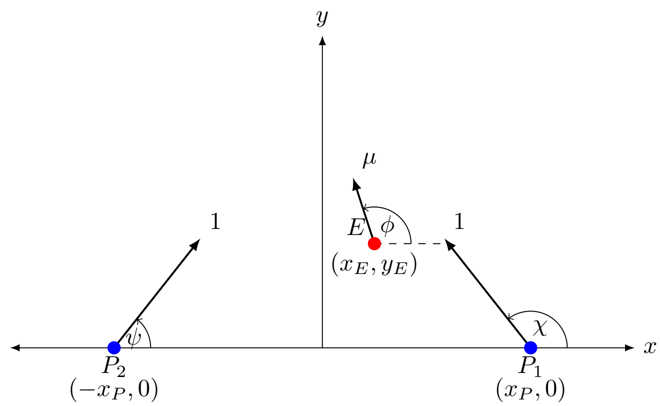

Fig. 1: Reduced dimension coordinate system.

The three-state nonlinear kinematics of the 2CFSG are given as [7]

$$\dot{x}_P = \frac{1}{2}(\cos\chi - \cos\psi), \tag{1}$$

$$\dot{x}_E = \mu\cos\phi - \frac{1}{2}\left(\cos\chi + \cos\psi\right) + \frac{1}{2}\frac{y_E}{x_P}\left(\sin\chi - \sin\psi\right), \tag{2}$$

$$\dot{y}_E = \mu\sin\phi - \frac{1}{2}\left(\sin\chi + \sin\psi\right) - \frac{1}{2}\frac{x_E}{x_P}\left(\sin\chi - \sin\psi\right), \tag{3}$$

with initial conditions

$$x_P(0) = x_{P_0}, \tag{4}$$

$$x_E(0) = x_{E_0}, \tag{5}$$

$$y_E(0) = y_{E_0}. \tag{6}$$

The angles χ, ψ, and ϕ represent the control actions of P_1, P_2, and E, respectively. In general, these angles are functions of time. The cost/payoff of the game is the time-to-capture, t_f, and the Value function is simply the minimax capture time,

$$V = \min_{\chi(t),\psi(t)} \max_{\phi(t)} \int_0^{t_f} 1 \, \mathrm{d}\tau \tag{7}$$

When the state of the system is in $R_{1,2}$, that is, the region in which the evader is captured simultaneously by both pursuers under optimal play, [7, Theorem 2] gives the players' optimal state feedback strategies and corresponding capture time as,

$$
\begin{aligned}
\sin\psi^* &= \frac{y_{E_0}+\sqrt{\mu^2 y_{E_0}^2+(1-\mu^2)\left(\mu^2 x_{P_0}^2-x_{E_0}^2\right)}}{\sqrt{(1-\mu^2)\left(x_{P_0}^2-x_{E_0}^2\right)+(1+\mu^2)\,y_{E_0}^2+2y_{E_0}\sqrt{\mu^2 y_{E_0}^2+(1-\mu^2)\left(\mu^2 x_{P_0}^2-x_{E_0}^2\right)}}},\\
\cos\psi^* &= \frac{\left(1-\mu^2\right)x_{P_0}}{\sqrt{(1-\mu^2)\left(x_{P_0}^2-x_{E_0}^2\right)+(1+\mu^2)\,y_{E_0}^2+2y_{E_0}\sqrt{\mu^2 y_{E_0}^2+(1-\mu^2)\left(\mu^2 x_{P_0}^2-x_{E_0}^2\right)}}},\\
\chi^* &= \pi-\psi^*,\\
\sin\phi^* &= \frac{1}{\mu}\frac{\mu^2 y_{E_0}+\sqrt{\mu^2 y_{E_0}^2+(1-\mu^2)\left(\mu^2 x_{P_0}^2-x_{E_0}^2\right)}}{\sqrt{(1-\mu^2)\left(x_{P_0}^2-x_{E_0}^2\right)+(1+\mu^2)\,y_{E_0}^2+2y_{E_0}\sqrt{\mu^2 y_{E_0}^2+(1-\mu^2)\left(\mu^2 x_{P_0}^2-x_{E_0}^2\right)}}},\\
\cos\phi^* &= -\frac{1}{\mu}\frac{\left(1-\mu^2\right)x_{E_0}}{\sqrt{(1-\mu^2)\left(x_{P_0}^2-x_{E_0}^2\right)+(1+\mu^2)\,y_{E_0}^2+2y_{E_0}\sqrt{\mu^2 y_{E_0}^2+(1-\mu^2)\left(\mu^2 x_{P_0}^2-x_{E_0}^2\right)}}},
\end{aligned} \tag{8}
$$

$$
t_f = \frac{1}{1-\mu^2}\sqrt{(1-\mu^2)\left(x_{P_0}^2-x_{E_0}^2\right)+(1+\mu^2)\,y_{E_0}^2+2y_E\sqrt{\mu^2 y_{E_0}^2+(1-\mu^2)\left(\mu^2 x_{P_0}^2-x_{E_0}^2\right)}}. \tag{9}
$$

Definition 1 (Singular configuration)**.** The singular configuration is any state (x_P, x_E, y_E) on the dispersal surface, that is,

$$\left\{(x_P, x_E, y_E)\ \middle|\ |x_E| < x_P,\ y_E = 0\right\}$$

III. Simple Motion and Simpler Motion

In Isaacs' "simple motion" paradigm the players control their respective headings while their speed is constant and known. But what if, for example the Evader, could control both his heading *and* his speed? We now investigate the interesting scenario where the Evader, by virtue of his ability to control his speed, opts to stay stationary. The Pursuers are cognizant of the Evader's capability to set his speed but of course are not aware of the Evader's decision. At the same time, having the Evader (E) reduce his speed to zero and thus stay in place is not entirely inconsistent with the simple motion paradigm: For example, suppose E goes around in a small circle at his constant speed μ near his initial position, and thus goes nowhere. In light of this, we now examine the case where E opts to stay put while P_1 and P_2 exercise their optimal pursuit strategies.

We consider the special case of a singular initial configuration where the three players are collinear and E is halfway between P_1 and P_2, the distance to the pursuers being x_{P_0}. The pursuers P_1 and P_2 exercise their respective optimal strategies as stipulated in (8), while the evader's control/heading $\phi(t)$ is a square wave that switches between 0 and π and whose period $T \to 0$ – this will leave the Evader in place, at the origin. Obviously, the pursuers are unaware of E's strategy/decision to stay put, that is, E's control time history $\phi(t)$ is not preannounced to the Pursuers; however, E's *nominal* speed, the problem parameter μ, is known. It turns out that the information of whether the Evader can or cannot reduce her speed is immaterial to the Pursuer. The pursuit-evasion game is analyzed in the reduced state space $\mathbb{R}^3_+$.

We now embark on calculating the time-to-capture t_f. Whenever P_1 and P_2 employ their respective optimal pursuit strategies $\psi^*(x_P, y(x_P, x_E, y_E))$ and $\chi^*(x_P, y(x_P, x_E, y_E))$ specified in (8) , the "closed loop" dynamics are

$$
\begin{aligned}
\dot{x}_P &= -\cos\psi^*, & x_P(0) &= x_{P_0}\\
\dot{x}_E &= \mu\cos\phi(t), & x_E(0) &= 0\\
\dot{y}_E &= -\sin\psi^* + \mu\sin\phi(t), & y_E(0) &= 0
\end{aligned}
$$

E's control $\phi(t)$ is now the square wave that rapidly switches between 0 and π because its period $T \to 0$. And since $\sin\phi(t) \equiv 0\ \forall\ t \geq 0$ and the average over time of the function $\cos\phi(t)$ is s.t.

$$\overline{\cos\phi(t)} \approx 0,$$

the "closed loop" dynamics are

$$
\begin{aligned}
\dot{x}_P &= -\cos\psi^*, & x_P(0) &= x_{P_0}\\
x_E &\approx 0 & &\\
\dot{y}_E &= -\sin\psi^*, & y_E(0) &= 0
\end{aligned}
$$

Since in the limit when the switching period $T \to 0$ the lateral displacement $x_E(t) \equiv 0$, from (8) we calculate

$$
\begin{aligned}
\sin\psi^* &= \frac{y_E + \mu\,\mathrm{sign}(y_E)\sqrt{y_E^2 + (1-\mu^2)x_P^2}}{\mu|y_E| + \sqrt{y_E^2 + (1-\mu^2)x_P^2}},\\
\cos\psi^* &= \frac{(1-\mu^2)x_P}{\mu|y_E| + \sqrt{y_E^2 + (1-\mu^2)x_P^2}}.
\end{aligned} \tag{10}
$$

Hence, the closed loop dynamics end up being

$$\dot{x}_P = -\frac{(1-\mu^2)x_P}{\mu|y_E| + \sqrt{y_E^2 + (1-\mu^2)x_P^2}}, \quad x_P(0) = x_{P_0}, \tag{11}$$

with

$$x_E(t) \equiv 0,$$

and

$$\dot{y}_E = -\frac{y_E + \mu\,\mathrm{sign}(y_E)\sqrt{y_E^2 + (1-\mu^2)x_P^2}}{|y_E| + \sqrt{y_E^2 + (1-\mu^2)x_P^2}}, \quad y_E(0) = 0, \tag{12}$$

and since $x_P(t)$ is monotonically decreasing in t we use (11) to calculate the time-to-capture

$$t_f = \frac{1}{1-\mu^2}\int_0^{x_{P_0}} \frac{\mu|y_E| + \sqrt{y_E^2 + (1-\mu^2)x_P^2}}{x_P}\,\mathrm{d}x_P, \tag{13}$$

where

$$\frac{\mathrm{d}y_E}{\mathrm{d}x_P} = \frac{1}{1-\mu^2}\frac{y_E + \mu\,\mathrm{sign}(y_E)\sqrt{y_E^2 + (1-\mu^2)x_P^2}}{x_P}, \quad y_E(x_{P_0}) = 0, \qquad x_{P_0} \geq x_P \geq 0. \tag{14}$$

From (12) we deduce that when $y_E > 0$, $\dot{y}_E < 0$, when $y_E < 0$, $\dot{y}_E > 0$ and when $y_E = 0$, $\dot{y}_E = 0$. Hence the solution of the y_E differential equation (14) is $y_E(t) \equiv 0$. Inserting $y_E = 0$ into (13) yields the time-to-capture

$$t_f = \frac{1}{\sqrt{1-\mu^2}}\, x_{P_0} \tag{15}$$

We could have arrived at the same conclusion more directly: (10) tell us that

$$\lim_{y_E\to 0,\ y_E>0} \sin\psi^* = \mu, \qquad \lim_{y_E\to 0,\ y_E<0} \sin\psi^* = -\mu$$

so when the switching frequency is high, that is, the period $T \to 0$,

$$\overline{\psi^*(t)} = 0, \tag{16}$$

$$\overline{\sin\psi^*(t)} = 0, \tag{17}$$

while at the same time

$$\lim_{y_E\to 0} \cos\psi^* = \sqrt{1-\mu^2}.$$

Also, from (10), when $y_E > 0$, $\psi^* > 0$ and when $y_E < 0$, $\psi^* < 0$.

Thus $\overline{\dot{y}_E(t)} = 0$ and therefore $y_E(t) \approx 0$, but $\cos\psi^*(t) \approx \sqrt{1-\mu^2}$. The closed loop dynamics are s.t. the pursuers are zigzagging along the x-axis until they reach the Evader at the origin. And because their speed along the x-axis is $\sqrt{1-\mu^2}$ while the initial distance to the origin is x_{P_0}, when the Evader stays put the time to reach the origin/time-to-capture is given by (15). By virtue of her not playing optimally but staying stationary at the origin, the Evader caused the optimally controlled Pursuers to zigzag and slow down. It is interesting to numerically investigate the case where the period T is small, but finite. We expect the speed of the Pursuers along the x-axis to remain the same but the Evader will be captured near the origin by just one of the two Pursuers while the time-to-capture will decrease.

It is also interesting to numerically investigate the case where the Evader opts to continuously modulate his heading according to $\phi(t) = \frac{\pi}{2}(1 + \sin\omega t)$ while the Pursuers employ their optimal strategies $\psi^*(x_P, y(x_P, x_E, y_E))$ and $\chi^*(x_P, y(x_P, x_E, y_E))$, whereupon the "closed loop" dynamics are

$$\dot{x}_P = -\cos\psi^*,$$
$$\dot{x}_E = \mu\cos(\frac{\pi}{2}(1+\sin\omega t)),$$
$$\dot{y}_E = -\sin\psi^* + \mu\sin\left(\frac{\pi}{2}(1+\sin\omega t)\right),$$

with initial conditions $x_P(0) = x_{P_0}$, $x_E(0) = x_{E_0}$, and $y_E(0) = y_{E_0}$. The parameter $\omega \gg 1$. To avoid aliasing, choose the integration time step/sampling rate $\Delta t = \frac{2\pi}{\omega n}$, with the integer $n \geq 5$. Now

$$x_E(t) = -\frac{\mu}{\omega}\int_0^{\omega t} \sin\left(\frac{\pi}{2}\sin(\tau)\right)\mathrm{d}\tau \tag{18}$$

and when $\omega \to \infty$, $x_E(t) \approx 0$. As before, the Pursuers' strategies are given by (10).

Given the singular initial configuration considered herein, the Evader's optimal strategy is to run along the orthogonal bisector of the $\overline{P_1P_2}$ segment and the Value of the game/time-to-capture

$$t_f = \frac{1}{\sqrt{1-\mu^2}} x_{P_0},$$

the same as in (15). Surprisingly, by deviating from the optimal strategy $\phi^*(x_P, y(x_P, x_E, y_E))$ specified in(8) and instead staying put, the Evader has *not* incurred a loss in his payoff and the Pursuers have not seen a decrease in their cost. Chattering control has allowed the Evader to achieve the same payoff as if he would have used his optimal strategy. However, the Evader's control is a valid control $\forall\ T > 0$, where T is the period of his square wave input function $\phi(t)$, but in the limit where $T \to 0$ the admissibility of the Evader's control might be questionable. So in reality, if $0 < T \ll 1$, the Evader will have slightly moved away from his initial position whereas our analysis is for the idealized case where the Evader has not moved at all. This discrepancy is responsible for the Evader not having the time-to-capture reduced and thus being penalized for deviating from her optimal strategy; at the same time, the Pursuers were not rewarded with a reduced time-to-capture, as one would have expected. This is attributable to the non physical control time history that the Evader was allowed to employ. But when the period T of E's maneuver is $T > 0$, numerical calculations will show that, as expected, his time-to-capture actually decreases.

It is thus interesting to consider the case where E modulates his heading according to

$$\phi(t) = \frac{\pi}{2}\left[1 + A\sin\left(\omega t - \frac{\pi}{2}\right)\right] \tag{19}$$

where a) $A = 0$, or b) $A > 0$. In case a) the Evader is employing his optimal strategy while in case b) the Evader is zigzagging around his optimal path. Can dithering gain the Evader an advantage, that is, increase his time-to capture above the "optimal" $t_f = x_{P_0}/\sqrt{1-\mu^2}$? The answer should be on the negative. The reader is referred to Section IV in the sequel.

Finally, had the Pursuers been informed ahead of time on the Evader's strategy of staying in place, the time-to-capture would have been reduced to $t_f = x_{P_0}$. However, the prior information of whether the Evader is or is not allowed to control his speed would not have changed the Pursuers' optimal strategy as specified in (8). Interestingly, Isaacs himself pondered whether controlling one's speed in the Homicidal Chauffeur differential game would make a difference. That controlling their speed affords no advantage to the players is corroborated by the verbatim quotation of footnote 1 of [10]:

> "Nothing is gained by bounding instead of fixing the speeds. Optimal strategies demand perpetual top speed."

In conclusion, the Pursuers do not know ahead of time that the Evader is stationary and when applying their optimal/saddle point strategy in the face of the evader's tardiness, they end up zigzagging along the x-axis until they reach the Evader. Interestingly, although the Evader deviated from his optimal/saddle point strategy, when the dither frequency is high, the Pursuers' cost did not decrease and neither did the evader's payoff decrease. As can happen in zero-sum games, this is an instance where unilaterally deviating from the saddle point strategy is not always detrimental; in deviating from the optimal/saddle point strategy and staying put at $(0, 0)$, the Evader can achieve the same capture time. The saddle point inequalities are not necessarily strict inequalities. Note, however, that standing still at $(0, 0)$ is *not* a saddle point strategy for the Evader as it is not robust to all possible Pursuer strategies. Had the Pursuers known ahead of time the Evader's strategy of staying put, they would have responded with the optimal strategy of running toward the Evader and the time-to-capture t_f would have decreased from the Value $x_{P_0}/\sqrt{1-\mu^2}$ to x_{P_0}.

To properly ascertain the demerits of the Evader's strategy of deviating from the optimal strategy and reverting to a stationary posture we need to consider states in the $R_{1,2}$ region of the state space where the Evader is in general position, that is, $x_{E_0} \neq 0$ and $y_{E_0} \neq 0$; without loss of generality assume $x_{E_0} > 0$, $y_{E_0} > 0$. When the P_1 and P_2 team employs its optimal strategy the frame x, y is *not* rotating and therefore by rapidly switching his control ϕ between 0 and π results in the Evader's fast movement from East to West and back; this indeed is tantamount to the Evader staying in place in the realistic plane. The dynamics in our three dimensional reduced state space are

$$\begin{aligned}
\dot{x}_P &= -\cos\psi*, \qquad x_P(0) = x_{P_0} \\
x_E(t) &= x_{E_0}, \\
\dot{y}_E &= -\sin\psi*, \qquad y_E(0) = y_{E_0}
\end{aligned}$$

and from (8)

$$\begin{aligned}
\sin\psi^* &= \frac{y_E + \sqrt{\mu^2 y_E^2 + (1-\mu^2)(\mu^2 x_P^2 - x_{E_0}^2)}}{\sqrt{(1-\mu^2)(x_P^2 - x_{E_0}^2) + (1+\mu^2)y_E^2 + 2y_E\sqrt{\mu^2 y_E^2 + (1-\mu^2)(\mu^2 x_P^2 - x_{E_0}^2)}}} \\
\cos\psi^* &= \frac{(1-\mu^2)x_P}{\sqrt{(1-\mu^2)(x_P^2 - x_{E_0}^2) + (1+\mu^2)y_E^2 + 2y_E\sqrt{\mu^2 y_E^2 + (1-\mu^2)(\mu^2 x_P^2 - x_{E_0}^2)}}}
\end{aligned} \tag{20}$$

Thus, the closed loop dynamics are

$$\begin{aligned}
\dot{x}_P &= \frac{-(1-\mu^2)x_P}{\sqrt{(1-\mu^2)(x_P^2 - x_{E_0}^2) + (1+\mu^2)y_E^2 + 2y_E\sqrt{\mu^2 y_E^2 + (1-\mu^2)(\mu^2 x_P^2 - x_{E_0}^2)}}}, \\
\dot{y}_E &= \frac{-y_E - \sqrt{\mu^2 y_E^2 + (1-\mu^2)(\mu^2 x_P^2 - x_{E_0}^2)}}{\sqrt{(1-\mu^2)(x_P^2 - x_{E_0}^2) + (1+\mu^2)y_E^2 + 2y_E\sqrt{\mu^2 y_E^2 + (1-\mu^2)(\mu^2 x_P^2 - x_{E_0}^2)}}}, \\
x_E(t) &= x_{E_0},
\end{aligned}$$

with initial conditions

$$x_P(0) = x_{P_0},$$
$$y_E(0) = y_{E_0}.$$

These dynamics are valid as long as the state stays in the state space region $R_{1,2}$, that is

$$x_P(t) \geq \frac{1}{\mu} x_{E_0}$$

Since x_P is monotonically decreasing, we integrate the non-autonomous differential equation

$$\frac{dy_E}{dx_P} = \frac{1}{1-\mu^2} \cdot \frac{y_E + \sqrt{\mu^2 y_E^2 + (1-\mu^2)(\mu^2 x_P^2 - x_{E_0}^2)}}{x_P},$$

with

$$y_E(x_{P_0}) = y_{E_0},$$
$$x_{P_0} \geq x_P \geq \frac{x_{E_0}}{\mu},$$

and obtain its solution $y_E(x_P)$, $x_{P_0} \geq x_P \geq \frac{x_{E_0}}{\mu}$. It is used to calculate the duration of the initial phase of the chase where both pursuers P_1 and P_2 participate:

$$t_{f_1} = \frac{1}{1-\mu^2} \int_{\frac{x_{E_0}}{\mu}}^{x_{P_0}} \frac{1}{x_P} \sqrt{(1-\mu^2)(x_P^2 - x_{E_0}^2) + (1+\mu^2) y_E^2 + 2 y_E \sqrt{\mu^2 y_E^2 + (1-\mu^2)(\mu^2 x_P^2 - x_{E_0}^2)}}\, \mathrm{d}x_P$$

The end game starts when the state leaves the $R_{1,2}$ region of the state space and at time t_{f_1} enters the R_1 region of the state space. At that point, we have $y_E(x_P) = y_E(x_{E_0}/\mu)$. In the end game, the Evader is captured solo by P_1 who heads straight toward the Evader and the duration of the end game

$$t_{f_2} = \sqrt{\left(\frac{1-\mu}{\mu}\right)^2 x_{E_0}^2 + \left[y_E\left(\frac{x_{E_0}}{\mu}\right)\right]^2},$$

so the duration of the chase is

$$t_f = t_{f_1} + t_{f_2}.$$

When $x_{E_0} \neq 0$ the chase always terminates in an endgame: If $x_{E_0} > 0$ the Evader is captured in the end game by P_1 and if $x_{E_0} < 0$ the Evader is captured in the end game by P_2. Under optimal play the duration of the chase is given by (9) and we expect that in general, by deviating from optimality and staying put, the Evader hastened his demise. While we have assumed the speeds of the players with simple motion are fixed, by reducing his speed, if this were at all possible, the Evader would gain no advantage. Obviously, the same applies to the Pursuers; at the same time, it is also true that a stationary Evader won't always be seriously penalized.

IV. Numerical Experimentation

In this section, several of the aforementioned scenarios, in which the Evader adopts a somewhat non-conventional strategy, are evaluated numerically. The simulations are performed by discretizing time and re-evaluating the agents' control policy at each time instant. The agents apply a fixed control over the duration of the time-step, that is $u(t) = u_{t_k}$ for $t_k \leq t \leq t_{k+1}$ and $k \in \mathbb{N}$, thus we have zero-order hold action. The paths of the agents are piecewise straight since their headings are constant for each segment. The point capture model described previously is replaced with a small, but finite, capture radius of $1e^{-}3$ about the pursuers. In the figures to follow, the Apollonius circles are drawn and the game-optimal intercept point is marked for $t = 0$. Note that the Apollonius circles and the game-optimal intercept point change over time as the agents move. Only in the case where all three agents head directly to the game-optimal intercept point does this point remain stationary in the realistic plane. For all of the scenarios, the Pursuers implement their optimal headings at each time step according to (8). The scenarios are simulated with $\mu = 0.5$, $v_P = 1$, and $x_P = 1$. The Evader's path is the red path, and the Pursuer paths are shown in blue in the following figures.

The first scenario is the case in which the Evader is on the y-axis and collinear with the Pursuers (near $(0,0)$) and implements a control where $\phi(t)$ is a square wave switching between 0 and π with period T ($T > 0$). One small caveat, here, is that we place the Evader just above the x-axis (at $y = 0.01$) in order for the initial game-optimal intercept point I to be unique.

Figures 2 and 3 contain the results of the simulation for two different choices for the Evader's switching period, T. In Fig. 2, $T = 0.1$ and the Evader's excursion from its start position is noticeable. The Pursuers zigzag in towards $x = 0$; the amplitude of their path is a function of the size of the timestep, which is 0.02 for this scenario. A smaller timestep will yield paths with smaller excursions from the x-axis. Note that for most of the scenario, the state lies in $R_{1,2}$, however, there is a short time just before capture in which the state enters R_1 wherein the Pursuers implement pure pursuit to finish out the game. Indeed this is one of the main reasons why $t_f < V$ (where V is the Value of the game): here, $t_f = 1$ and $V = 1.16$. Thus the evader's performance was worsened under this policy compared to if he had gone to I, as expected. In Fig. 3, the switching period is reduced to $T = 0.02$ and we see that the capture time $t_f = 1.13$ is closer to V. As $T \to 0$ we expect $t_f \to V$, however, for smaller and smaller T

we must have smaller and smaller integration time steps Δt. Otherwise, for finite Δt, the value of t_f may actually exceed V, which can be attributed to the phenomenon of aliasing in digital signal processing.

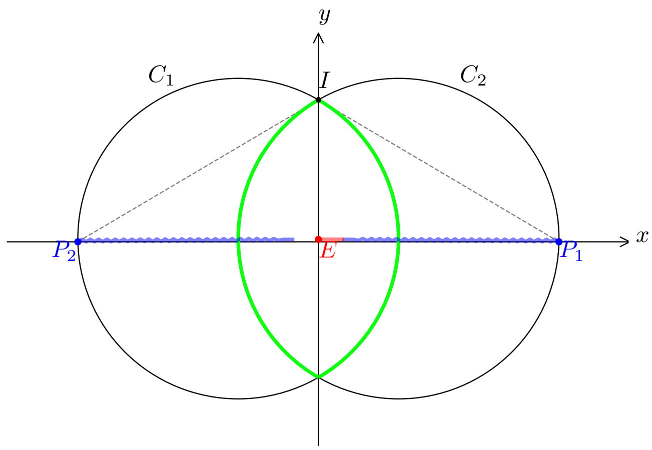

Fig. 2: Evader implements square wave policy with $\phi(t) = 0$ or π and $\Delta t = 0.02$. Based on initial conditions, the Value of the game is $V = 1.16$. Here, $T = 0.10$, $t_f = 1.00$.

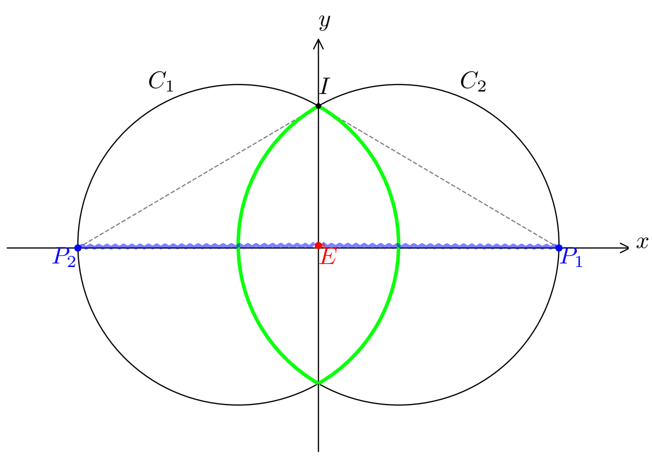

Fig. 3: Same scenario as previous figure but with $T = 0.02$, $t_f = 1.13$.

The second scenario has the same initial conditions as the first, but the Evader implements the policy in which $\phi(t)$ oscillates about the optimal heading ($\phi^* = \pi/2$) according to (19).

Figure 4 contains the results for this scenario. The Evader's non-optimal trajectory induces curvature in the Pursuers' trajectories. The Evader's capture time $t_f = 1.14$ is less than what it would have been if he had traveled straight for the point I, as expected.

Lastly, we investigate the case in which the Evader stands perfectly still but at a point that is away from the dispersal surface which occurs at $(x_E, y_E) = (0, 0)$.

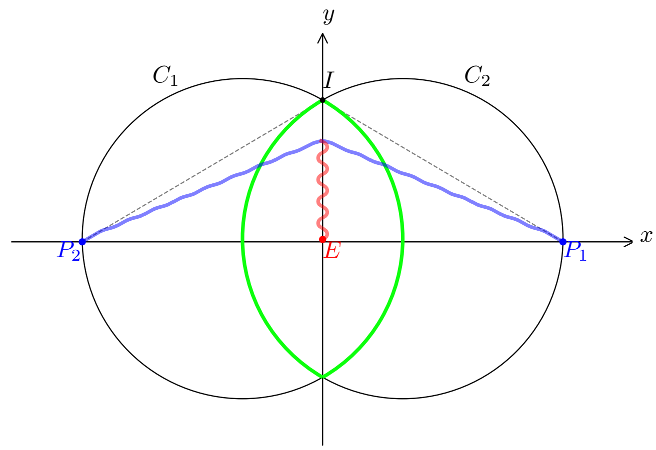

Fig. 4: Evader implements policy where $\phi(t)$ oscillates about the optimal ϕ^*. $\Delta t = 0.001$, $A = 0.7$, $\omega = 50$, $t_f = 1.14$; $V = 1.16$.

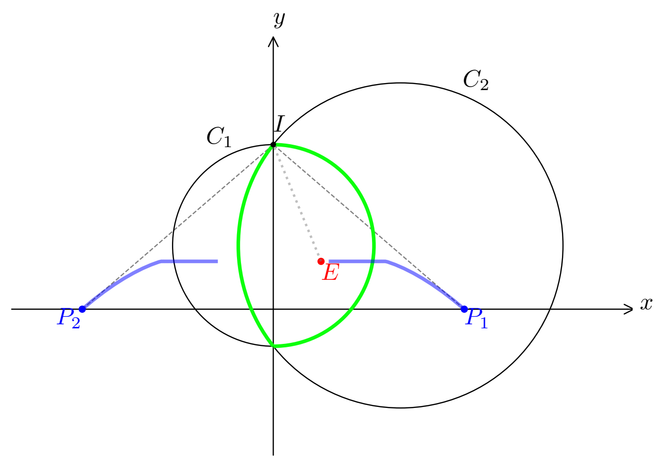

Fig. 5: Evader stands still at $(x_E, y_E) = (0.25, 0.25)$, $\Delta t = 0.001$, $t_f = 0.83$; $V = 1.32$.

Figure 5 contains the results of the simulation. Note, because the initial conditions are different in this scenario, the Value of the game is also different. As discussed in the preceding section, this scenario results in two phases of play for the Pursuers. In the first phase, from $0 < t < t_{f_1}$, the Pursuers take a curved path which deviates away from the game-optimal trajectories. From the simulation, $t_{f_1} \approx 0.576$. The knee in the Pursuers' curves occurs when the three agents become collinear, after which the Pursuers zigzag towards the Evader until t_{f_1}. At t_{f_1}, the state of the system enters R_1, and thus we have $x_E/\mu = x_P = 0.5$ [see 7, Theorem 1]. Already, we know the y position of P_1 is 0.25 since t_{f_1} occurs after the knee, in this case. Thus t_{f_2} is the time needed for P_1 to cover a distance of 0.25. By standing still away from $(0, 0)$, in the relative frame, the evaders performance is *much* worse than if he had chosen to head for I. In summary, for each of the three scenarios considered, the Evader cannot improve his capture time above the game-optimal Value by, e.g., dithering or staying put. This might not be so if more pursuers join the fight.

V. Conclusion

In this paper, we have carefully considered the singular surface present in the Two Cutters and Fugitive Ship game posed by Isaacs. Our purpose in investigating the dispersal surface was to determine whether the Evader could exploit the presence of the singularity. We showed that, in discrete-time with piece-wise constant headings, the Pursuers' control will chatter when faced with an Evader who stands still on or near the dispersal surface. In the limit as we shrank the timestep to zero, the effect of the Pursuers' zigzagging was that they approached the Evader at reduced speed. Nevertheless, the accumulation of many infinitesimal performance losses of the Pursuers resulted in the Pursuers reaching the Evader precisely at the time dictated by the Value of the game. Thus, the Evader was not penalized for standing still in such a configuration. It is generally known that the regular/primary solutions, e.g. the ones corresponding to the canonical Two Cutters and Fugitive Ship pursuit strategy, are not always sufficient to fully describe optimal play in differential games – hence the famous quote of Isaacs in the Introduction. We have shown a prime example: although the Evader did not *lengthen* its life by standing still, it was able to achieve the Value of the game using a strategy other than the canonical strategy. One implication is in two-pursuer, multiple-evader scenarios with capture in succession: an intelligent evader equipped with this knowledge may be able to aid a fellow evader by choosing its own, out of the way, capture location while not sacrificing any remaining life. Another implication is that specialized pursuit strategies may be required in the vicinity of singularities to prevent exploitation by the evader(s).

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