Two-on-One Pursuit*

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I. Introduction

Pursuit and evasion has a strong aerospace connotation. In Surface-to-Air Missile (SAM) engagements the standard procedure is to fire two SAMs to intercept a threat. It is conventional wisdom that the probability of kill of a SAM is $P_K = 0.8$ and therefore when two SAMs are launched the probability of kill is enhanced and is $P_K = 0.96$. Note, however, that it is herein tacitly assumed that the SAMs are "independent" statistically speaking. Their effectiveness could be improved if they were cooperatively guided. Thus, in this paper the foundational pursuit-evasion differential game in the Euclidean plane where two Pursuers P_1 and P_2 *cooperatively* chase an Evader E, is considered. The three players are holonomic, the speeds of the Pursuers each being greater than that of the Evader. We are interested in point capture by either one, or both, P_1 and/or P_2. The payoff of E and the cost of the P_1 & P_2 team is time-to-capture. Thus, Isaacs' classical "Two Cutters and a Fugitive Ship" differential game is revisited. Interestingly, the Two Cutters and Fugitive Ship pursuit game was posed by Hugo Steinhaus back in 1925 – his original paper was reprinted in 1960 in [1]*. The solution, sans a proof, of the differential game is presented in Isaacs' ground breaking book [2, Example 6.8.3, pp. 148-149] where the players' optimal strategies were derived using a geometric method. Since then, several others have investigated the game as well as other closely related games. In [3], the game of one fast pursuer against two evaders is solved. Ganebny et al. consider a two pursuer one evader game on a line [4]. Most recently, the two pursuer one evader scenario (in two dimensions) was investigated in [5] wherein evader strategies are derived for the case where the Evader *knows* the Pursuers' strategies. In [6] a proof of the optimality of the three players' strategies proposed by Isaacs is undertaken. Reference [7] analyzes the two-pursuer one-evader game with a finite capture radius,

*The views expressed in this paper are those of the authors and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the United States Government

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*Hugo Steinhaus, was a contemporary of Borel and Von Neumann who are credited with laying the foundations of game theory. Borel and Von Neumann mainly considered static games, a.k.a. games in normal form, while referring to dynamic games as games in extensive form, believing that dynamic games can be easily transformed to static games. The requirement of time consistency/subgame perfectness in dynamic games came to the attention of game theorists only in the seventies. From the outset, Steinhaus was certainly attune to thinking about dynamic games, a.k.a. differential games.

making use of the costate equilibrium dynamics to solve a boundary value problem backwards in time. The extension of this game for any number of pursuers is explored in [8] wherein open loop strategies are proposed with no proof of optimality. We also note the presence of a significant body of literature on both the two-on-one and multi-on-one differential games with one or more of the following features: fixed duration, cost/payoff defined as terminal miss distance, various kinematic/dynamic models (e.g., inertial vs. non-inertial, bounded acceleration, bounded velocity, etc.), integral constraints, and/or a superior evader [9–15].

In this paper, some geometric features, perhaps overlooked by Isaacs, but with a bearing on extensions, are addressed: The state space regions where pursuit devolves into Pure Pursuit (PP) by either P_1 or P_2, or into a pincer movement pursuit by the P_1 & P_2 team who cooperatively capture the evader, are characterized. Thus, in this paper, a complete solution of the Game of Kind is provided. Furthermore, in this paper, a three dimensional reduced state space for analyzing the two-on-one pursuit-evasion differential game is introduced and the players' state feedback strategies as well as the Value function are explicitly derived.

The paper is organized as follows. The geometric method employed by Isaacs to solve the Two Cutters and Fugitive Ship differential game is expounded on in Section II. In Section III a three-states reduced state space reformulation of the Two-on-One pursuit-evasion differential game is introduced and Isaacs' geometric method is employed to yield the players' optimal state feedback strategies and the game's Value function in closed form. Furthermore, the state space regions where either one of the pursuers captures the evader and the state space region where both pursuers cooperatively and isochronously capture the evader are characterized, thus solving the Game of Kind. Possible extensions are also discussed in Section III. Conclusions are presented in Section IV.

II. The Geometric Method

We assume that the fast pursuers P_1 and P_2 have equal speed, which we normalize to 1. The problem parameter is the speed of the evader E which is $0 \leq \mu < 1$.

There are three players in the Euclidean plane so the realistic state space is obviously $\mathbb{R}^6$, however the state space could be reduced to $\mathbb{R}^4$ by collocating the origin of a non rotating (x, y) Cartesian frame at E's instantaneous position. Since the players are holonomic, the dynamics A matrix is 0 – there are no dynamics. This, and the fact that the performance functional is the time-to-capture, yields a Hamiltonian s.t. the costates are all constant. This suggests that the optimal flow field might consist of straight line trajectories. Hence geometry might come into play. To obtain the Two Cutters and Fugitive Ship differential game's solution, Isaacs employed the geometric concept of an Apollonius circle to delineate the Boundary of a Safe Region (BSR) for the Evader: In pursuit-evasion differential games an Apollonius circle is constructed based on the E-P separation and the speed ratio $\mu < 1$. The Apollonius circle concept is conducive to the geometric solution of the Two Cutters and Fugitive Ship differential game, as will be demonstrated in the sequel. For a more thorough treatment of the Apollonius circle, see [16]; we include the main features here for reference. The Apollonius circle radius and circle center (along the ray $\overrightarrow{PE}$) are given by,

$$\rho = \frac{\mu}{1-\mu^2}\overline{PE}, \tag{1}$$

$$x_O = \frac{\mu^2}{1-\mu^2}\overline{PE}, \qquad y_O = 0. \tag{2}$$

A. Isaacs' Geometric Solution

We first present the solution of the Two Cutters and Fugitive Ship differential game in the realistic plane using the geometric method. Two Apollonius circles, C_1, whose foci are at E and P_1 and the Apollonius circle C_2, whose foci are are at E and P_2, feature in this game. E is in the interior of both Apollonius disks but the two Apollonius circles might or might not intersect. Concerning the calculation of the points of intersection, if any, of the Apollonius circles C_1 and C_2: Subtracting the equation of circle C_1 from the equation of circle C_2 yields a linear equation in two unknowns, say, X and Y. One can thus back out Y as a function of X and insert this expression into one of the circle equations, thus obtaining a quadratic equation in X: The calculation of the two points of intersection of the Apollonius circles C_1 and C_2 boils down to the solution of a quadratic equation. The Apollonius circles intersect *iff* the quadratic equation has real solutions, in other words, the discriminant of the quadratic equation is positive. When the discriminant of the quadratic equation is negative we are automatically notified that the Apollonius circles don't intersect, and because E is in the interior of both Apollonius disks, we conclude that one of the Apollonius disks is contained in the interior of the second Apollonius disk. If $\rho_2 > \rho_1$, which is the case *iff* E is closer to P_1 than to P_2 – see (1) – the circle C_2 is discarded, and vice versa. The geometry is illustrated in Fig. 1.

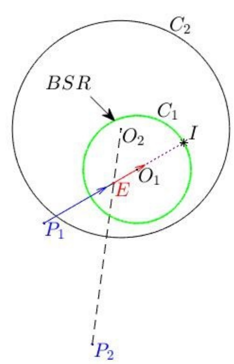

Figure 1 One Cutter Action

When the Apollonius circles don't intersect, the pursuer associated with the outer Apollonius circle is irrelevant to the chase. This is so because the configuration is s.t. should P_1 employ PP and E run for his life, player P_2 cannot reach E before the latter is captured by P_1 because he is too far away from the P_1/E engagement, or is too slow to close in and join the fight. This renders player P_2 irrelevant. As far as the geometric method is concerned, the Apollonius disk associated with player P_1 is then contained in the interior of the bigger Apollonius disk associated with player P_2, as illustrated in Fig. 1. In this case the pursuer P_1 on which the inner Apollonius circle is based will singlehandedly capture the evader: He will optimally employ PP while the Evader runs for his life and will be captured at I; the game with two pursuers devolved to the simple pursuit-evasion game with one pursuer and one evader where P_1 employs PP and E runs away from P_1. Similarly, if the Apollonius disk associated with P_2 is contained in the interior of the bigger Apollonius disk associated with player P_1, player P_2 will employ PP while E runs for his life; P_1 is then redundant.

The interesting case, considered in [2], where the Apollonius circles intersect is illustrated in Fig. 2.

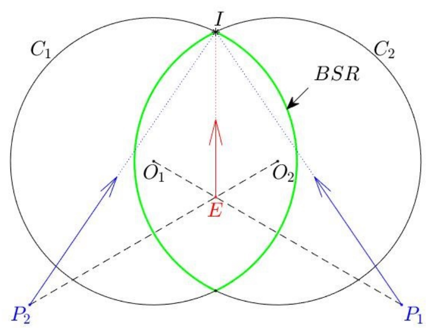

Figure 2 Solution of Two Cutters and Fugitive Ship Game

Since there are two pursuers, similar to Figure 6.8.5 in [2], a lens-shaped BSR, delineated in green, is formed by the intersection of the two Apollonius circles. To calculate the aim point I which is one of the two points where the Apollonius circles C_1 and C_2 intersect requires solving a quadratic equation; the quadratic equation has two real solutions and among the two points of intersection of the Apollonius circles, the aim point I is the point farthest from E. Thus, E heads toward the most distant point I on the BSR, and so do P_1 and P_2. Both pursuers P_1 and P_2 will be active and cooperatively and isochronously capture the evader at point I – see Fig. 2.

When the discriminant of the quadratic equation is zero the quadratic equation has a repeated real root. Geometrically this means that one of the Apollonius circles is tangent from the inside to the second Apollonius circle. The following holds.

Proposition 1. Assume the Apollonius circles C_1 and C_2 are tangent, that is, the discriminant of the quadratic equation vanishes. The aim point of the three players is then the circles' point of tangency, say T, that is, $I = T$, *iff* the three players E, P_1 and P_2 are collinear and E is sandwiched between P_1 and P_2. □

When the Apollonius circles C_1 and C_2 are tangent and their point of tangency T is s.t. $T = I$, the points P_2, T, O_1, E, O_2 and P_1 are collinear and both pursuers employ PP to isochronously capture the evader. This is illustrated in

Fig. 3.

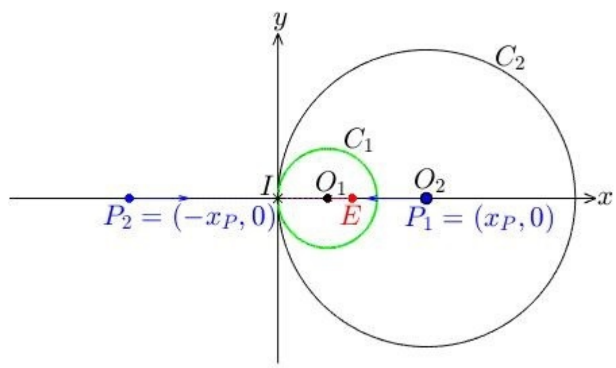

Figure 3 PP by P_1 *and* P_2

Note however that when, as above, P_1, P_2 and E are collinear and E is sandwiched between P_1 and P_2, but the Apollonius circles intersect, E will break out – see Fig. 4.

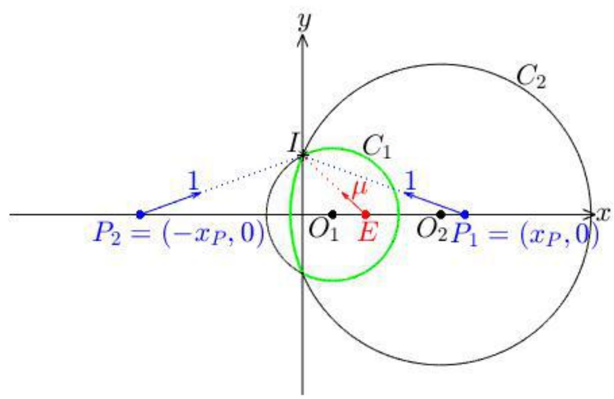

Figure 4 Breakout of E

If the Apollonius circles C_1 and C_2 are tangent, however E is not on the segment $\overline{P_1P_2}$, the players' aim point I is not the circles' point of tangency T: If the tangent Apollonius circles are s.t. the Apollonius circle C_1 is contained in the Apollonius disk formed by the Apollonius circle C_2, optimal play then consists of the active player being P_1 and employing PP while E runs away from P_1 and player P_2 is redundant; and if the Apollonius circle C_2 is contained in the Apollonius disk formed by the Apollonius circle C_1, optimal play then consists of the active player being P_2 and employing PP while E runs away from P_2, and now player P_1 is redundant; the circles' point of tangency T plays no role here. This should alert us to the fact that even though the Apollonius circles intersect at their point of tangency, that is, $C_1 \cap C_2 \neq \emptyset$ and $T \in C_1 \cap C_2$, the players' aim point $I \ni C_1 \cap C_2$.

III. Geometric Solution in Reduced State Space

Similar to Isaacs' treatment of the Homicidal Chauffeur differential game, it is beneficial to analyze the Two Cutters and Fugitive Ship differential game in a reduced state space. The dimension of the Two Cutters and Fugitive Ship game's state space can be reduced to three using a non-inertial, rotating reference frame, by pegging the x-axis to P_1 and P_2's instantaneous positions. The y-axis is the orthogonal bisector of the $\overline{P_1 P_2}$ segment. In this rotating (x, y) reference frame the states are E's x and y-coordinates (x_E, y_E) and the x-position x_P of P_1. In this reduced state space the y-coordinate of P_1 will always be 0 and the position of P_2 will be $(-x_P, 0)$. Without loss of generality we assume $x_E \geq 0$ and $y_E \geq 0$. The rotating reference frame (x, y) is shown overlaid on the realistic plane (X, Y) in Fig. 5 where the P_1, E and P_2 players' headings χ, ϕ and ψ are also indicated. Without loss of generality, the rotating reference frame (x, y) is initially aligned with the inertial frame (X, Y).

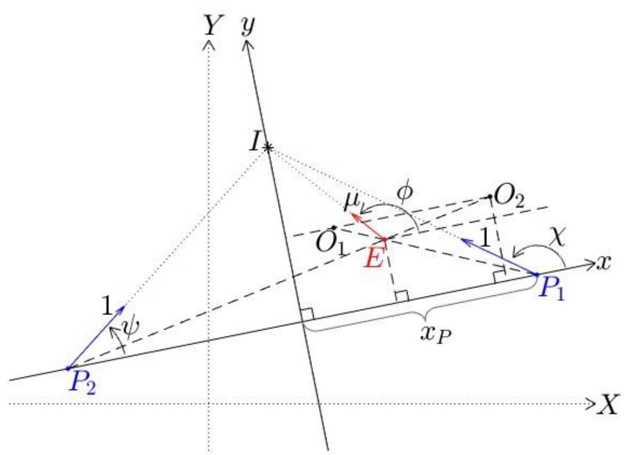

Figure 5 Rotating Reference Frame

Using the rotating reference frame (x, y), the state space of the Two Cutters and Fugitive Ship differential game is reduced to the first quadrant of $\mathbb{R}^3$, that is, the set

$$\mathbb{R}^3_1 \equiv \{(x_P, x_E, y_E) \mid x_P \geq 0, y_E \geq 0\}$$

Symmetry allows us to confine our attention to the case where $x_E \geq 0$, that is, the state will evolve in the positive orthant of $\mathbb{R}^3$, that is, in

$$\mathbb{R}^3_+ = \{(x_P, x_E, y_E) \mid x_P \geq 0, x_E \geq 0, y_E \geq 0\},$$

where the three-state nonlinear dynamics of the Two Cutters and Fugitive Ship differential game are

$$\dot{x}_P = \frac{1}{2}(\cos\chi - \cos\psi), \qquad x_P(0) = x_{P_0} \tag{3}$$

$$\dot{x}_E = \mu\cos\phi - \frac{1}{2}(\cos\chi + \cos\psi) + \frac{1}{2}\frac{y_E}{x_P}(\sin\chi - \sin\psi), \qquad x_E(0) = x_{E_0} \tag{4}$$

$$\dot{y}_E = \mu\sin\phi - \frac{1}{2}(\sin\chi + \sin\psi) - \frac{1}{2}\frac{x_E}{x_P}(\sin\chi - \sin\psi), \qquad y_E(0) = y_{E_0}. \tag{5}$$

A. Game of Kind in Reduced State Space

The solution of the Game of Kind determines which pursuer actually captures the evader under optimal play: P_1, P_2, or simultaneous capture by both pursuers. More specifically, the Game of Kind partitions the state space into three regions which dictate the outcome of the differential game under optimal play. The solution of the Game of Kind in the reduced state space (x_P, x_E, y_E) using the geometric method proceeds as follows.

We have two Apollonius circles: C_1 is based on the instantaneous positions of E and P_1, and C_2 is based on the instantaneous positions of E and P_2. In the (x, y) frame, see Fig. 4 and (2), the center O_1 of the Apollonius circle C_1 is at

$$x_{O_1} = \frac{1}{1-\mu^2}\left(x_E - \mu^2 x_P\right), \qquad y_{O_1} = \frac{1}{1-\mu^2} y_E.$$

Similarly, the center O_2 of the Apollonius circle C_2 is at

$$x_{O_2} = \frac{1}{1-\mu^2}\left(x_E + \mu^2 x_P\right), \qquad y_{O_2} = \frac{1}{1-\mu^2} y_E.$$

Thus, using (1), the equation of the Apollonius circle C_1 is

$$\left[x - \frac{1}{1-\mu^2}\left(x_E - \mu^2 x_P\right)\right]^2 + \left(y - \frac{1}{1-\mu^2} y_E\right)^2 = \frac{\mu^2}{\left(1-\mu^2\right)^2}\left[(x_E - x_P)^2 + y_E^2\right], \tag{6}$$

and the equation of the Apollonius circle C_2 is

$$\left[x - \frac{1}{1-\mu^2}\left(x_E + \mu^2 x_P\right)\right]^2 + \left(y - \frac{1}{1-\mu^2} y_E\right)^2 = \frac{\mu^2}{\left(1-\mu^2\right)^2}\left[(x_E + x_P)^2 + y_E^2\right]. \tag{7}$$

In the (x, y) reference frame the y-coordinate of the C_1 and C_2 Apollonius circles' centers is the same and therefore the distance d between the circles' centers is

$$d = x_{O_2} - x_{O_1}$$
$$= \frac{2\mu^2}{1-\mu^2} x_P$$

Hence, because the radii of the Apollonius circles are s.t. $\rho_1 < \rho_2$ *iff* $x_E > 0$, the Apollonius circles C_1 and C_2 intersect *iff*

$$d + \rho_1 > \rho_2,$$

that is,

$$2\mu x_P + d_1 > d_2.$$

In other words, the inequality holds

$$2\mu x_P > \sqrt{(x_P + x_E)^2 + y_E^2} - \sqrt{(x_P - x_E)^2 + y_E^2}$$

which yields the algebraic condition: The Apollonius circles C_1 and C_2 intersect *iff*

$$\mu^2 y_E^2 + \left(1 - \mu^2\right)\left(\mu^2 x_P^2 - x_E^2\right) \geq 0. \tag{8}$$

In light of this, the reduced state space $\mathbb{R}_1^3$ is partitioned as follows.

$$\mathbb{R}_1^3 = R_1 \cup R_2 \cup R_{1,2}$$

During optimal play in R_1, E is captured solely by P_1 while P_2 is redundant, in R_2 E is captured solely by P_2 while P_1 is redundant, while in $R_{1,2}$, E is isochronously captured by P_1 and P_2. At this point it appears that things stand as follows. If condition (8) does not hold and $x_E > 0$ the state is in R_1, where E is captured solo by P_1. If condition (8) does not hold and $x_E < 0$ the state is in R_2, where E is captured solo by P_2: From a kinematic point of view, the state is in R_1 if Collision Course (CC) guidance won't allow P_2 to capture E who is running away from P_1, before P_1, using Pure Pursuit (PP), captures E. Similarly, the state is in R_2 if CC guidance won't allow P_1 to capture E who is running away from P_2, before P_2, using PP, captures E. As far as geometry is concerned, let D_i denote the disk which corresponds to the Apollonius circle C_i, $i = 1, 2$. In view of the discussion from above, it would appear that the set R_1 is characterized by $D_1 \subset D_2$ – see Fig. 1; similarly, the set R_2 is characterized by $D_2 \subset D_1$, and if condition (8) holds – see Fig. 2 where the Apollonius circles C_1 and C_2 intersect – one might then be inclined to think that the state is in $R_{1,2}$, so that during optimal play E is isochronously captured by P_1 and P_2. And as far as the characterization of the sets R_1 and R_2 is concerned, since $x_E \geq 0$ implies $\rho_1 \leq \rho_2$, the disk D_2 cannot be contained in the disk D_1, so either $D_1 \subset D_2$ or the Apollonius circles C_1 and C_2 intersect. The geometric condition

$$D_1 \subset D_2 \;\Rightarrow\; d + \rho_1 < \rho_2$$

lets us recover the algebraic condition (8):

$$C_1 \cap C_2 \neq \emptyset \;\Leftrightarrow\; d + \rho_1 > \rho_2 \;\Leftrightarrow\; \mu^2 y_E^2 + \left(1 - \mu^2\right)\left(\mu^2 x_P^2 - x_E^2\right) > 0,$$

as expected. The algebraic condition (8) delineates the set in $\mathbb{R}_+^3$,

$$\mathcal{K}_1 = \left\{(x_P, x_E, y_E) \mid x_P \geq 0, x_E \geq 0, \mu^2 y_E^2 + \left(1 - \mu^2\right)\left(\mu^2 x_P^2 - x_E^2\right) < 0\right\}.$$

This is a cone whose x_E cross sections are arcs of ellipses.When the state is in the interior of the elliptical cone $\mathcal{K}_1$ or in its projection onto the plane $y_E = 0$, $D_1 \subset D_2$ and so E is captured by P_1 only. Thus, one is inclined to set $R_1 \equiv \mathcal{K}_1$. Similarly, when the state is in the interior of the elliptical cone

$$\mathcal{K}_2 = \left\{(x_P, x_E, y_E) \mid x_P \geq 0, x_E \leq 0, \mu^2 y_E^2 + \left(1 - \mu^2\right)\left(\mu^2 x_P^2 - x_E^2\right) < 0\right\}$$

or in its projection onto the plane $y_E = 0$, $D_2 \subset D_1$ and so E is captured by P_2 only; the set $\mathcal{K}_2$ is the mirror image of the cone $\mathcal{K}_1$ about the plane $x_E = 0$ and one is inclined to set $R_2 \equiv \mathcal{K}_2$. The boundary of the elliptical cone $\mathcal{K}_1$ is the set of states s.t. the Apollonius circle C_1 is contained in the Apollonius disk formed by the bigger circle C_2 and is tangent to the Apollonius circle C_2; similarly, the boundary of the elliptical cone $\mathcal{K}_2$ is the set of states s.t. the Apollonius circle C_2 is contained in the Apollonius disk formed by the bigger circle C_1 and is tangent to the Apollonius circle C_1. When the state is on the boundary of the elliptical cones $\mathcal{K}_1$ or $\mathcal{K}_2$ the Apollonius circles C_1 and C_2 are tangent, say, at point T. According to Proposition 1, the players' aim point I is the point of tangency T of the Apollonius circles *iff* $y_E = 0$ and the tangent to the Apollonius circles at $T = I$ is the orthogonal bisector of the segment $\overline{P_1 P_2}$; and from (8) we deduce $x_E = \mu x_P$; E is then isochronously captured by P_1 and P_2 who employ PP – as illustrated in Fig. 3. Note that if $x_E = 0$, condition (8) holds, so the quarter plane $\{(x_P, x_E, y_E) \mid x_P \geq 0, x_E = 0, y_E \geq 0\} \subset R_{1,2}$ and E is isochronously captured by P_1 and P_2. Obviously E is also isochronously captured by P_1 and P_2 when $x_P = 0$. And so far, it would appear that during "optimal" play, when the state is outside the elliptical cones $\mathcal{K}_1$ and $\mathcal{K}_2$ where the inequality (8) holds, that is, the state is in what appears to be $R_{1,2}$, E will be isochronously captured by the P_1&P_2 team. Thus, at first blush it would appear that (8) characterizes the set $R_{1,2}$. However, as will become apparent in the sequel, although in the set $R_{1,2}$ the inequality (8) holds, it also holds in subsets of R_1 and R_2: condition (8) does *not* characterize the set

$R_{1,2}$. We must properly characterize the state space regions R_1, R_2 and $R_{1,2}$ in $\mathbb{R}^3_1$. The inequality (8) does not provide the answer and it will be replaced by an alternative condition.

In this respect, consider the following. In Fig. 1 let the points E and P_2 be fixed while point P_1 is moved in a clockwise direction, keeping the $P_1 - E$ distance d_1 constant so that the Apollonius circles C_1 and C_2 will eventually intersect, whereupon the inequality (8) will hold. The radius ρ_1 of the Apollonius circle C_1 is kept constant while it is approaching the Apollonius circle C_2 from the inside. The Apollonius circle C_1 first meets the Apollonius circle C_2 tangentially and if the segment $\overline{P_1E}$ rotates some more clockwise, the circles start intersecting. When this initially happens, the point I in Fig. 1 is still in the interior of the disk formed by the Apollonius circle C_2. Thus, although the Apollonius circles intersect and condition (8) holds, E nevertheless flees toward point I with P_1 in hot pursuit, as if the configuration would have been as illustrated in Fig. 1 where the Apollonius circle C_1 is in the interior of the Apollonius disk formed by the Apollonius circle C_2; it is only when point I on the extension of the segment $\overline{EO_1}$ meets the Apollonius circle C_2 and then exists the disk formed by the Apollonius circle C_2, that both pursuers, P_1 *and* P_2 cooperatively and isochronously capture E in a pincer movement maneuver. Thus, although the Apollonius circles do intersect, it nevertheless might be the case that neither one of their two points of intersection is the players' aim point I, and as before, only one of the pursuers is active while the Evader runs for his life from the active pursuer. The BSR then has the shape of a thick lens and the Evader's and the active pursuer's aim point I is the point on the thick lens-shaped BSR which is farthest away from E – it is on the circumference of the smaller Apollonius circle, on its diameter that runs though E, while at the same time it is in the *interior* of the Apollonius disk formed by the bigger Apollonius circle. The critical configuration where point $I \in C_2$ is illustrated in Fig. 6.

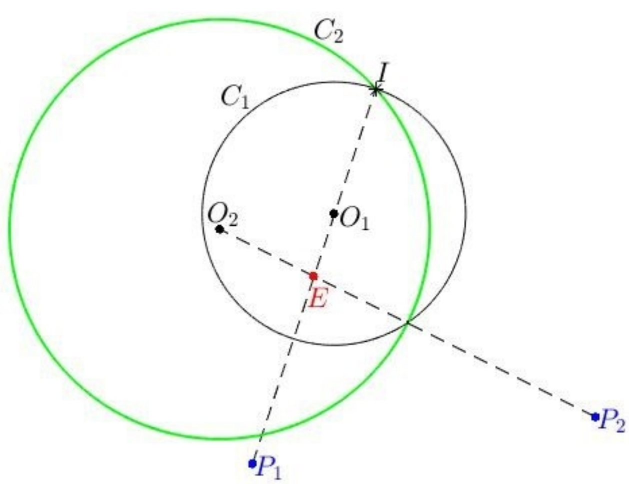

Figure 6 Critical Configuration

Since, without loss of generality, we have assumed $x_E \geq 0$ and $y_E \geq 0$, our universe of discourse will be confined to the positive orthant of $\mathbb{R}^3$, $\mathbb{R}^3_+$.

Theorem 1. During optimal play the Evader is singlehandedly captured in PP by P_1 if the state is in the set R_1; the set R_1 is the wedge formed by the quarter planes $\{(x_P, x_E, y_E) \mid x_P = 0, x_E \geq 0, y_E \geq 0\}$ and $\{(x_P, x_E, y_E) \mid x_E = \mu x_P, x_P \geq 0, y_E \geq 0\}$. The Evader is singlehandedly captured in PP by P_2 if the state is in the set R_2; the set R_2 is the mirror image of R_1 about the plane $x_E = 0$. The Evader is cooperatively and isochronously captured by P_1 and P_2 if the state is in the set

$$R_{1,2} = \{(x_P, x_E, y_E) \mid -\mu x_P \leq x_E \leq \mu x_P,\ x_P \geq 0, y_E \geq 0\}.$$

Proof. To obtain a correct algebraic characterization of the sets R_1, R_2 and $R_{1,2}$ which will supersede condition (8), proceed as follows. Calculate the (x, y) coordinates of the critical point I on the circumference of the Apollonius circle C_1 which is antipodal to E, as shown in Fig. 6 – see Fig. 7:

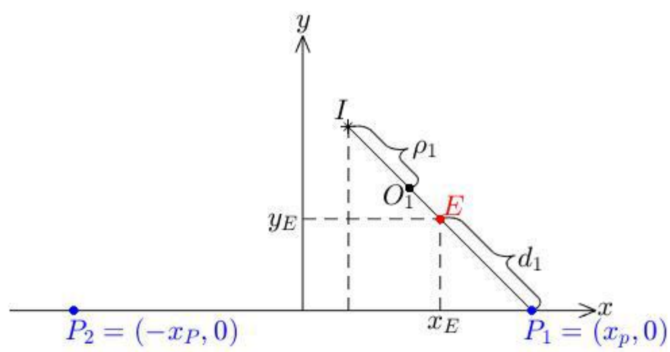

Figure 7 Point I

We have

$$\frac{x_P - x_I}{x_P - x_E} = \frac{\rho_1 + \overline{EO}_1 + d_1}{d_1}, \qquad \frac{y_I}{y_E} = \frac{\rho_1 + \overline{EO}_1 + d_1}{d_1},$$

where

$$\overline{EO}_1 = \frac{\mu^2}{1-\mu^2} d_1, \qquad \rho_1 = \frac{\mu}{1-\mu^2} d_1.$$

Hence,

$$x_I = \frac{1}{1-\mu}(x_E - \mu x_P), \qquad y_I = \frac{1}{1-\mu} y_E. \tag{9}$$

By construction, $I \in C_1$ and I is the critical aim point if in addition $I \in C_2$. To find the points of intersection (x_I, y_I)

of the circles C_1 and C_2 boils down to the solution of a quadratic equation:

$$x_I = 0, \qquad y_I = \frac{y_E + \sqrt{\mu^2 y_E^2 + \left(1-\mu^2\right)\left(\mu^2 x_P^2 - x_E^2\right)}}{1-\mu^2} \tag{10}$$

Combining (9) and (10) we obtain the result

$$x_E = \mu x_P,$$

and the solution of the Game of Kind is as stated in this Theorem. □

The cones $\mathcal{K}_1$ and $\mathcal{K}_2$ and/or condition (8) have no role to play here. The Apollonius circles C_1 and C_2 intersect if $-\mu x_P \leq x_E \leq \mu x_P$.

Remark. Proposition 1 is a corollary of Theorem 1.

In summary, the reduced state space of the Two Cutters and Fugitive Ship differential game is the first quadrant of $\mathbb{R}^3$, that is, $\mathbb{R}^3_1 = \{(x_P, x_E, y_E) \mid x_P \geq 0, y_E \geq 0\}$. The state space $\mathbb{R}^3_1$ is symmetric about the plane $x_E = 0$; the region R_1 (and $\mathcal{K}_1$) reside in the positive orthant $\mathbb{R}^3_+$. Since point capture is desired, the terminal set in the R_1 subset of the $\mathbb{R}^3_+$ state space is the straight line $\{(x_P, x_E, y_E) \mid x_E = x_P, x_P \geq 0, y_E = 0\}$ and the terminal set in the $R_{1,2}$ subset of the state space is the origin.

B. Game of Degree in Reduced State Space

1. Game in R_1 and R_2

In R_1 the active pursuer P_1 employs PP while the evader runs for his life. The actions of pursuer P_2 do not affect the outcome of the game and so, for exclusively illustrative purposes, we stipulate that P_2 mirrors the control of P_1. This ensures that the (x, y) frame won't rotate – it would just slide upward along the Y axis of the realistic plane, which then coincides with the y-axis. The optimal trajectories in R_1 are the family of straight lines

$$x_P(t) = x_{P_0} + \frac{x_{E_0} - x_{P_0}}{\sqrt{(x_{P_0} - x_{E_0})^2 + y_{E_0}^2}}\, t,$$

$$x_E(t) = x_{E_0} + \mu \frac{x_{E_0} - x_{P_0}}{\sqrt{(x_{P_0} - x_{E_0})^2 + y_{E_0}^2}}\, t,$$

$$y_E(t) = y_{E_0} - (1-\mu) \frac{y_{E_0}}{\sqrt{(x_{P_0} - x_{E_0})^2 + y_{E_0}^2}}\, t.$$

The state $y_E(t)$ is monotonically decreasing and when parameterized by y_E, the optimal trajectories in R_1 are the family of straight lines

$$\begin{aligned} x_P &= \frac{1}{1-\mu}\left(\frac{x_{P_0}-x_{E_0}}{y_{E_0}}\, y_E + x_{E_0} - \mu x_{P_0}\right), \\ x_E &= \frac{1}{1-\mu}\left(\mu\frac{x_{P_0}-x_{E_0}}{y_{E_0}}\, y_E + x_{E_0} - \mu x_{P_0}\right). \end{aligned}$$

These trajectories terminate in the plane $y_E = 0$, on the straight line $x_P = x_E$. The optimal flow field in R_1 consists of the family of straight line trajectories from above, which terminate on the straight line $\{(x_P, x_E, y_E) \mid x_E = x_P, y_E = 0\}$. Similar considerations apply to R_2 where the active pursuer is P_2. The optimal flow field in R_2 is a mirror image of the optimal flow field in R_1.

When $x_p = 0$, P_1 and P_2 are collocated. The half plane $\{(x_P, x_E, y_E) \mid x_P = 0, y_E \geq 0\} \subset R_1 \cup R_2$.

2. Game in $R_{1,2}$

If the state is in

$$R_{1,2} = \{(x_P, x_E, y_E) \mid -\mu x_P \leq x_E \leq \mu x_P, x_P \geq 0, y_E \geq 0\},$$

E will be isochronously captured by the $P_1\&P_2$ team. The players' optimal headings are given in the next Theorem.

Theorem 2. The players' optimal headings are constant in both the (x, y) and (X, Y) frames and they are given by

$$\begin{aligned} \sin\psi^* &= \frac{y_{E_0} + \sqrt{\mu^2 y_{E_0}^2 + \left(1-\mu^2\right)\left(\mu^2 x_{P_0}^2 - x_{E_0}^2\right)}}{\sqrt{\left(1-\mu^2\right)\left(x_{P_0}^2 - x_{E_0}^2\right) + \left(1+\mu^2\right) y_{E_0}^2 + 2y_{E_0}\sqrt{\mu^2 y_{E_0}^2 + \left(1-\mu^2\right)\left(\mu^2 x_{P_0}^2 - x_{E_0}^2\right)}}}, \\ \cos\psi^* &= \frac{\left(1-\mu^2\right) x_{P_0}}{\sqrt{\left(1-\mu^2\right)\left(x_{P_0}^2 - x_{E_0}^2\right) + \left(1+\mu^2\right) y_{E_0}^2 + 2y_{E_0}\sqrt{\mu^2 y_{E_0}^2 + \left(1-\mu^2\right)\left(\mu^2 x_{P_0}^2 - x_{E_0}^2\right)}}}, \\ \chi^* &= \pi - \psi^*, \\ \sin\phi^* &= \frac{1}{\mu}\frac{\mu^2 y_{E_0} + \sqrt{\mu^2 y_{E_0}^2 + \left(1-\mu^2\right)\left(\mu^2 x_{P_0}^2 - x_{E_0}^2\right)}}{\sqrt{\left(1-\mu^2\right)\left(x_{P_0}^2 - x_{E_0}^2\right) + \left(1+\mu^2\right) y_{E_0}^2 + 2y_{E_0}\sqrt{\mu^2 y_{E_0}^2 + \left(1-\mu^2\right)\left(\mu^2 x_{P_0}^2 - x_{E_0}^2\right)}}}, \\ \cos\phi^* &= -\frac{1}{\mu}\frac{\left(1-\mu^2\right) x_{E_0}}{\sqrt{\left(1-\mu^2\right)\left(x_{P_0}^2 - x_{E_0}^2\right) + \left(1+\mu^2\right) y_{E_0}^2 + 2y_{E_0}\sqrt{\mu^2 y_{E_0}^2 + \left(1-\mu^2\right)\left(\mu^2 x_{P_0}^2 - x_{E_0}^2\right)}}}. \end{aligned} \tag{11}$$

The initial state $(x_{P_0}, x_{E_0}, y_{E_0})$ can momentarily be viewed as the current state and as such, Eqs. (11) are explicit state feedback "optimal" strategies, as provided by the geometric method; the attendant Value function is given by

$$t_f = \frac{1}{1-\mu^2}\sqrt{\left(1-\mu^2\right)\left(x_{P_0}^2 - x_{E_0}^2\right) + \left(1+\mu^2\right)y_{E_0}^2 + 2y_E\sqrt{\mu^2 y_{E_0}^2 + \left(1-\mu^2\right)\left(\mu^2 x_{P_0}^2 - x_{E_0}^2\right)}} \tag{12}$$

Proof. Since the $\triangle\, P_1P_2I$ in Fig. 2 is isosceles, the aim point $I = (0, y)$ is obtained upon setting $x = 0$ in (6) or (7), which yields a quadratic equation in y. The discriminant of the quadratic equation is positive *iff* the Apollonius circles C_1 and C_2 intersect, which is the case *iff* condition (8) holds and is certainly the case if $\mu x_P \leq x_E \leq \mu x_P$, whereupon

$$y = \frac{1}{1-\mu^2}\left[y_E + \text{sign}(y_E)\sqrt{\mu^2 y_E^2 + \left(1-\mu^2\right)\left(\mu^2 x_P^2 - x_E^2\right)}\right],$$

where the function

$$\text{sign}(x) \equiv \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0, \end{cases}$$

so

$$y_I = \frac{1}{1-\mu^2}\left[y_E + \text{sign}(y_E)\sqrt{\mu^2 y_E^2 + \left(1-\mu^2\right)\left(\mu^2 x_P^2 - x_E^2\right)}\right]. \tag{13}$$

Using the geometric method, the players' optimal state feedback strategies in $R_{1,2}$ are explicitly given by

$$\sin\psi^* = \frac{y_I}{\sqrt{x_P^2 + y_I^2}}, \qquad \cos\psi^* = \frac{x_P}{\sqrt{x_P^2 + y_I^2}}, \tag{14}$$

$$\sin\chi^* = \frac{y_I}{\sqrt{x_P^2 + y_I^2}}, \qquad \cos\chi^* = -\frac{x_P}{\sqrt{x_P^2 + y_I^2}}, \tag{15}$$

$$\sin\phi^* = \frac{y_I - y_E}{\sqrt{(y_I - y_E)^2 + x_E^2}}, \qquad \cos\phi^* = -\frac{x_E}{\sqrt{(y_I - y_E)^2 + x_E^2}}, \tag{16}$$

and the time-to-capture/Value function is

$$V(x_P, x_E, y_E) = \sqrt{x_P^2 + y_I^2}, \tag{17}$$

where the function $y_I(x_P, x_E, y_E)$ is given by (13).

When the initial state $(x_{P_0}, x_{E_0}, y_{E_0}) \in R_{1,2}$ and P_1, P_2 and E play optimally, the closed loop dynamics are

$$\begin{aligned} \dot{x}_P &= G x_P, \qquad x_P(0) = x_{P_0}, \\ \dot{x}_E &= G x_E, \qquad x_E(0) = x_{E_0}, \\ \dot{y}_E &= G y_E, \qquad y_E(0) = y_{E_0},\ 0 \le t, \end{aligned} \tag{18}$$

where

$$G = -\frac{(1-\mu^2)}{\sqrt{(1-\mu^2)\left(x_P^2 - x_E^2\right) + (1+\mu^2)\, y_E^2 + 2 y_E \sqrt{\mu^2 y_E^2 + (1-\mu^2)\left(\mu^2 x_P^2 - x_E^2\right)}}}.$$

The solution of the system (18) of strongly nonlinear differential equations is simply

$$\begin{aligned} x_P(t) &= \left(1 - \frac{t}{t_f}\right) x_{P_0}, \\ x_E(t) &= \left(1 - \frac{t}{t_f}\right) x_{E_0}, \\ y_E(t) &= \left(1 - \frac{t}{t_f}\right) y_{E_0}, \qquad 0 \le t \le t_f \end{aligned} \tag{19}$$

where t_f is given by (12). Inserting (19) into (14)–(16) we obtain the players' constant headings in both the (x, y) and (X, Y) frames. □

When the geometric method is applied and P_1 and P_2 play "optimally", from (11) we deduce that in the (x, y) frame the headings of P_1 and P_2 are mirror images of each other: $\chi^* = \pi - \psi^*$. Therefore, the (x, y) frame does not rotate and the players' headings are constant also in the (inertial) (X, Y) frame of the realistic plane. Hence, in the realistic plane, the "optimal" trajectories are straight lines. Since initially the rotating (x, y) frame is aligned with the (X, Y) frame of the realistic plane, the y-axis stays aligned with the Y-axis while the x-axis stays parallel to the X-axis moving in the upward direction at a constant speed. Therefore the "optimal" trajectories are also straight lines in the (x, y) frame. Thus, when the state feedback strategies (11) synthesized using the geometric method are applied, the closed loop system's "optimal" flow field in the $R_{1,2}$ region of the reduced state space consists of the family of straight line trajectories (19) which converge at the origin. Moreover, this flow field, which was produced by the geometric method, *covers* the $R_{1,2}$ region of the reduced state space – this, by construction.

The following extensions are of interest. The cutters' speeds need not be equal. Furthermore, it is interesting to consider the case where the speed of just one of the two cutters, say P_1, is higher than the speed of the fugitive ship while the speed of P_2 is equal to the speed of the fugitive ship. In this case, upon employing the geometric method, the Apollonius circle which is based on E and P_2 devolves into the orthogonal bisector of the segment $\overline{EP_2}$. It makes sense to also stipulate that the cutters P_1 and P_2 are endowed with circular capture sets with radii $l_1 > 0$ and $l_2 > 0$

respectively. In this case the elegant Apollonius circles will be replaced by Cartesian ovals and the boundary separating the R_1, R_2, and $R_{1,2}$ regions of the state space won't be planar and will be replaced by a more complex surface.

IV. Conclusion

In this paper Isaacs' Two Cutters and Fugitive Ship differential game has been revisited. The solution of the Game of Kind is provided, that is, the partition of the state space into regions where under optimal play just one of the pursuers captures the evader, and the state space region where both pursuers cooperatively capture the target, has been characterized. The closed form solution of the Game of Degree has been obtained using Isaacs' geometric method.

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