Circular Target Defense Differential Games*

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***Abstract*—In this paper, the problem of guarding a circular target wherein the Defender(s) is constrained to move along its perimeter is posed and solved using a differential game theoretic approach. Both the one-Defender and two-Defender scenarios are analyzed and solved. The mobile Attacker seeks to reach the perimeter of the circular target, whereas the Defender(s) seeks to align itself with the Attacker, thereby ending the game. In the former case, the Attacker wins, and the Attacker and Defender play a zero-sum differential game where the payoff/cost is the terminal angular separation. In the latter case, the Defender(s) wins, and the Attacker and Defender play a zero-sum differential game where the cost/payoff is the Attacker's terminal distance to the target. This formulation is representative of a scenario in which the Attacker inflicts damage on the target as a function of its terminal distance. The state-feedback equilibrium strategies and Value functions for the Attacker-win and Defender(s)-win scenarios are derived for both the one- and two-Defender cases, thus providing a solution to the Game of Degree. Analytic expressions for the separating surfaces between the various terminal scenarios are derived, thus providing a solution to the Game of Kind. An alternative game is formulated and solved in the case of Attacker win wherein the Attacker seeks to minimize time to reach the target.**

***Index Terms*—Differential games, optimal control, differential equations**

I. Introduction

The problem of guarding a target has many important applications in real-world defense scenarios. One example is protection of a building's perimeter from mobile infiltrators, which may be considered to be people, ground vehicles, air vehicles, or even certain types of munitions. Isaacs considered such a target guarding problem in his seminal work on differential game theory [1, see Example 1.9.2]. There, the static target was a convex area. Recently, there has been an interest in the defense of a mobile target, usually represented by a point or disk (e.g. [2]).

We pose and analyze the target guarding problem wherein the Defender (D) is constrained to move along the (static) circular target perimeter and the mobile Attacker (A) moves with simple motion. This problem is an instance of the perimeter defense problem presented in [3]–[5]; these works establish strategies for individual agents as well as teams of Attackers and Defenders for targets of arbitrary convex shape. One significant difference in the formulation presented here is that we consider the game to terminate when either A reaches the target (A wins), or D becomes aligned with A (D wins). The latter scenario may be thought of as the Defender being able to neutralize the Attacker (at a distance) with a highly directional weapon.

Because the perimeter is a circle and the Defender is constrained to move along the circle, this problem has a strong connection to the *Lady in the Lake* (or swimmer and bear; c.f. [6]) differential game wherein the pursuer runs along the shoreline of a lake (the circle) to try and catch the evader who must swim to the shore from inside the lake to escape [7]–[9]. Here, however, we are essentially analyzing the *Lady Outside the Lake* game with the agents' roles reversed. The cost functional, in the case that the Attacker can reach the perimeter, is identical to the *Lady in the Lake* game. Note that a similar Defender model is used in [10] (i.e., where the Defender(s) is (are) constrained to move along a circle), however, there, the Attacker initially begins inside the circle and seeks to escape. The overall solution approach is also similar – we utilize a classical differential game approach [1], [9] to obtain the Value function, equilibrium strategies and *Game of Kind* (or barrier) surface [10].

In general, target guarding may be thought of a special case of reach-avoid games wherein one player seeks to reach a goal configuration while avoiding some undesirable configuration (such as capture). Much work has been done on reach-avoid games ranging from numerical computation of the reachable sets for each player to extensions to teams of players via decomposition [11]–[13]. This work is most closely related to some recent works which build up the solution to two-on-one scenarios using a rigorous differential game formulation with the one-on-one solution as a basis. For example, [14] derives the analytical barrier between Attacker and Defenders winning for a reach-avoid game which takes place inside a rectangular domain. A similar result has been obtained for a blocking game where two Defenders seek to prevent an Attacker from reaching a line segment [15]. Finally, several recent works have analyzed reach-avoid games inside a circular domain but with freely moving Defenders [16], [17]. The utility of these rigorous analytical results have been highlighted in task allocation schemes which are capable of handling teams of many agents [18].

This paper contains the following contributions: (i) the one-on-one Attacker-win and Defender-win scenarios are formulated and solved rigorously using a differential game theoretic approach, verifying the saddle-point equilibrium status of strategies existing in the literature [3]; (ii) analytic expressions for the Value functions are derived for both one-on-one

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scenarios; (iii) the two-Defender, one-Attacker scenarios are formulated and the equilibrium strategies and Value functions are derived; (iv) the entire state space is partitioned based on all of the different terminal scenarios, and analytic expressions for the separating surfaces are derived; (v) an alternative scenario in which the Attacker seeks to reach the target in minimum time is solved. The emphasis is on the analysis and proof methods, which are based on differential game theory, in comparison to the geometric methods used previously [3]. Sections II and III cover the one- and two-Defender cases, respectively. In each of those sections, both the Attacker-win and Defender(s)-win scenarios are formulated and solved. This work is an extension of [19], which includes expanded proofs and analysis of the problem of minimum time penetration (Sections II-D and III-D). Section IV concludes the work.

II. One Defender

This section formulates the target guarding problem wherein the Defender (D) is constrained to move along the circular target perimeter and the Attacker (A) moves in the plane with simple motion. Figure 1 shows the local coordinate system (black) used in much of the analysis to appear, as well as the global (inertial) (x, y)-coordinate system (green). The

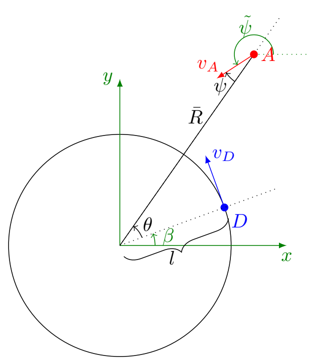

Fig. 1. Circular perimeter patrol with one Defender and one Attacker.

following assumptions are made on the problem setup:

Assumption 1. The players' speeds are such that $0 < v_A \leq v_D$, where v_A and v_D are the speeds of the Attacker and Defender, respectively.

Assumption 2. The initial separation angle is such that $\theta(t_0) = \theta_0 \in [0, \pi)$.

Assumption 3. The initial Attacker distance is such that $R(t_0) > 1$ – that is, A begins outside the target circle.

Assumption 2 will be lifted after the equilibrium strategies have been derived and the symmetry (and attendant singularity) identified. The (dimensional) kinematics, based on Fig. 1 are

$$\bar{f}(\bar{\mathbf{x}}, \bar{u}, \bar{t}) = \dot{\bar{\mathbf{x}}} = \begin{bmatrix} \dot{\bar{R}} \\ \dot{\bar{\theta}} \\ \dot{\bar{\beta}} \end{bmatrix} = \begin{bmatrix} -v_A \cos\psi \\ \frac{v_A}{\bar{R}} \sin\psi - \frac{v_D}{l} \\ \frac{v_D}{l} \end{bmatrix}, \qquad (1)$$

where $\bar{\theta} \in [-\pi, \pi]$ is the angle of A's position w.r.t. D and $\beta \in [0, 2\pi]$ represents the rotation of D about the circle's center w.r.t. a global (x, y)-plane. With the following definitions,

$$R \equiv \frac{\bar{R}}{l}, \quad t \equiv \frac{v_{D_{\max}}}{l}\bar{t}, \quad u_D \equiv \frac{v_D}{v_{D_{\max}}}, \quad \nu = \frac{v_A}{v_{D_{\max}}},$$

where $v_{D_{\max}}$ is the maximum Defender speed and the speed ratio $0 < \nu \leq 1$, the kinematics in (1) are non-dimensionalized:

$$f(\mathbf{x}, u, t) = \dot{\mathbf{x}} = \begin{bmatrix} \dot{R} \\ \dot{\theta} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -\nu\cos\psi \\ \nu\frac{1}{R}\sin\psi - u_D \\ u_D \end{bmatrix}. \qquad (2)$$

The Defender control lies in the range $u_D \in [-1, 1]$, and the Attacker control lies in the range $\psi \in [-\pi, \pi]$. Note θ and $\bar{\theta}$ are equivalent, but their time derivatives differ due to the scaling of time.

We define the *Game of Kind* as the question of whether Attacker can reach the perimeter ($R \to 1$) with non-zero terminal separation angle (Attacker 'wins') or the Defender can drive $\theta \to 0$ before the Attacker reaches the perimeter (Defender 'wins'). The subscript f refers to conditions at termination (e.g., t_f is the terminal time). In the following sections, the surface separating these two cases is derived and a *Game of Degree* is specified and solved for each case.

Note that if $v_A > v_D$, the Attacker need only come within some distance $l < \bar{R} < l\frac{v_A}{v_D}$ wherein the Attacker has the control authority to force $\theta \to \pi$. Similarly, when $v_A \leq v_D$, if at some point $\theta = 0$ the game is over because the Defender has sufficient control authority to keep $\theta = 0$ regardless of the Attacker's control. We assume that if $\theta_f = 0$ the Defender has successfully intercepted the Attacker and thwarted its attack. We refer to the question of whether the Attacker wins (i.e. $\theta_f > 0$) or the Defender wins ($\theta_f = 0$) as the *Game of Kind*.

A. Defender Win Scenario

In this section we are concerned with the *Game of Degree* which takes place when D is able to drive $\theta \to 0$ before A can reach the target. Here, the initial condition of the system lies in the region $\mathscr{R}_D$, which is the region of win for the Defender (see (28)). In this case, it is sensible for the agents to play a zero-sum game over the cost functional

$$J_d = \Phi_d(\mathbf{x}_f, t_f) = -R_f, \qquad (3)$$

where the subscript f denotes conditions at termination. The negative sign in (3) is present so that the Defender is the minimizing player and the Attacker is the maximizing player. That is, the Attacker seeks to get as close as possible to $R_f = 1$ and the Defender seeks to maximize the terminal distance. We refer to this game as the *Game of Distance*, and denote it with subscript d, in general. The Value of the game, if it exists, is the saddle-point equilibrium of the cost functional over state-feedback strategies

$$V_d = \min_{u_D(\cdot)} \max_{\psi(\cdot)} J_d = \max_{\psi(\cdot)} \min_{u_D(\cdot)} J_d. \qquad (4)$$

The terminal constraint for the *Game of Distance* is

$$\phi_d\left(\mathbf{x}_f, t_f\right) = \theta_f = 0. \tag{5}$$

The final time, t_f, is the first time for which $\theta(t) = 0$. Thus, the Terminal Surface is defined as the set of states satisfying (5)

$$\mathscr{T}_d = \left\{\mathbf{x} \;\mid\; R > 1 \text{ and } \theta = 0\right\}. \tag{6}$$

Assumptions 1 and 2 are retained for this analysis.

1) First Order Necessary Conditions for Optimality: We carry out the analysis according to a classical differential game approach [1] [9, c.f., Ch. 8, Thm 2, and §8.5]. The kinematics remain unchanged from the previous analysis; the Hamiltonian for the *Game of Distance* is

$$\mathscr{H}_d = -\sigma_R \nu \cos\psi + \sigma_\theta\left(\nu\frac{1}{R}\sin\psi - u_D\right) + \sigma_\beta u_D, \tag{7}$$

where $\sigma \equiv \begin{bmatrix}\sigma_R & \sigma_\theta & \sigma_\beta\end{bmatrix}^\top$ is the adjoint vector for the *Game of Distance*. The Hamiltonian is a separable function of the controls u_D and ψ, and thus *Isaacs' condition* [1], [9] holds:

$$\min_{u_D}\max_{\psi}\mathscr{H}_d = \max_{\psi}\min_{u_D}\mathscr{H}_d, \qquad \forall\mathbf{x},$$

where $u_D \in [-1, 1]$ and $\psi \in [-\pi, \pi]$. The equilibrium adjoint dynamics are given by

$$\dot{\sigma}_R = -\frac{\partial\mathscr{H}_d}{\partial R} = \nu\sigma_\theta\frac{1}{R^2}\sin\psi, \tag{8}$$

$$\dot{\sigma}_\theta = -\frac{\partial\mathscr{H}_d}{\partial\theta} = 0, \tag{9}$$

$$\dot{\sigma}_\beta = -\frac{\partial\mathscr{H}_d}{\partial\beta} = 0. \tag{10}$$

The terminal adjoint values are obtained from the transversality condition [20, pg. 89]

$$\begin{aligned}\sigma^\top(t_f) &= \frac{\partial\Phi_d}{\partial\mathbf{x}_f} + \eta\frac{\partial\phi_d}{\partial\mathbf{x}_f}\\ &= \begin{bmatrix}-1 & 0 & 0\end{bmatrix} + \eta\begin{bmatrix}0 & 1 & 0\end{bmatrix}\\ &\qquad \sigma_{R_f} = -1\\ &\implies \sigma_{\theta_f} = \eta\\ &\qquad \sigma_{\beta_f} = 0,\end{aligned} \tag{11}$$

where η is an additional adjoint variable whose value will be determined later in the analysis. Therefore, with (9)–(11), the following hold

$$\sigma_\theta(t) = \eta, \qquad \forall t \in [t_0, t_f] \tag{12}$$

$$\sigma_\beta(t) = 0, \qquad \forall t \in [t_0, t_f]. \tag{13}$$

Once again, since $\sigma_\beta(t) = 0$ for all $t \in [t_0, t_f]$, the state component β has no effect on the equilibrium trajectory or the equilibrium control strategies. The terminal Hamiltonian satisfies [20]

$$\mathscr{H}_d(t_f) = -\frac{\partial\Phi_d}{\partial t_f} - \eta\frac{\partial\phi_d}{\partial t_f} = 0, \tag{14}$$

and $\frac{\mathrm{d}\mathscr{H}_d}{\mathrm{d}t} = 0$, so $\mathscr{H}_d(t) = 0$ for all $t \in [t_0, t_f]$.

The equilibrium control actions of the Attacker and Defender maximize and minimize (7), respectively: $\mathscr{H}_d^* = \max_\psi\min_{u_D}\mathscr{H}_d$. In order to maximize (7) (with (12)), the vector $\begin{bmatrix}\cos\psi & \sin\psi\end{bmatrix}$ must be parallel to the vector $\begin{bmatrix}\sigma_R & \frac{\eta}{R}\end{bmatrix}$, giving

$$\cos\psi^* = \frac{-\sigma_R}{\sqrt{\sigma_R^2 + \frac{\eta^2}{R^2}}}, \qquad \sin\psi^* = \frac{\eta}{R\sqrt{\sigma_R^2 + \frac{\eta^2}{R^2}}}. \tag{15}$$

If $\eta < 0$, this implies $\sin\psi^* < 0$ due to (15). However, this would mean the Attacker has a component of its motion that points *towards* the Defender due to Assumption 2 (see, e.g., Fig. 1). Thus, it must be the case that $\eta > 0$. In order to minimize (7) (with (12)), the Defender's control must satisfy

$$u_D^* = \operatorname{sign}\eta = 1, \tag{16}$$

since $\eta > 0$.

Substituting the equilibrium controls, (15) and (16), into the Hamiltonian, (7), and evaluating at final time with (11) and (14) gives

$$\begin{aligned}\mathscr{H}_d^*(t_f) = 0 &= \frac{\nu\sigma_{R_f}^2}{\sqrt{\sigma_{R_f}^2 + \frac{\eta^2}{R_f^2}}} + \frac{\nu\eta^2}{R_f^2\sqrt{\sigma_{R_f}^2 + \frac{\eta^2}{R_f^2}}} - \eta\\ &\implies \eta = \pm\nu R_f\sqrt{\frac{1}{R_f^2 - \nu^2}}.\end{aligned}$$

Since $\eta > 0$, we have

$$\eta = \nu R_f\sqrt{\frac{1}{R_f^2 - \nu^2}}. \tag{17}$$

2) Solution Characteristics: An expression for σ_R is obtained by considering the Hamiltonian at a general time, making the same substitutions as before, with the additional substitution of (17):

$$\begin{aligned}\mathscr{H}_d^*(t) = 0 &= \nu\sqrt{\sigma_R^2 + \frac{\eta^2}{R^2}} - \eta\\ \implies \sigma_R &= \pm\sqrt{\frac{\eta^2}{\nu^2} - \frac{\eta^2}{R^2}}\\ &= \pm\frac{R_f}{R}\sqrt{\frac{R^2 - \nu^2}{R_f^2 - \nu^2}}.\end{aligned}$$

Since $\sigma_{R_f} < 0$ (due to (11)) and $\dot{\sigma}_R > 0$ (due to (8) with (15) and $\eta > 0$) it must be that $\sigma_R(t) < 0$ for all $t \in [t_0, t_f]$, thus

$$\sigma_R = -\frac{R_f}{R}\sqrt{\frac{R^2 - \nu^2}{R_f^2 - \nu^2}}. \tag{18}$$

The retrograde equilibrium kinematics (denoted by $\mathring{\mathbf{x}}^*$, where $\mathring{\mathbf{x}}^* = -\dot{\mathbf{x}}^*$) can be obtained by substituting the equilibrium controls, (15) and (16), along with the adjoints, (12), (13), and (18), into (2) which yields

$$\mathring{R}^* = \nu\sqrt{1 - \frac{\nu^2}{R^2}}, \qquad \mathring{\theta}^* = 1 - \frac{\nu^2}{R^2}, \tag{19}$$

with the following boundary conditions

$$R(t_f) > 1, \qquad \theta(t_f) = 0. \tag{20}$$

Note that both $\mathring{R}$ and $\mathring{\theta}$ are monotonically increasing according to (19). Consider the differential equation obtained by dividing the equations in (19)

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = \frac{\nu}{\sqrt{1-\frac{\nu^2}{R^2}}}$$
$$\implies \nu \left[\sqrt{\frac{R^2}{\nu^2}-1} + \sin^{-1}\left(\frac{\nu}{R}\right)\right]_{R_f}^{R} = \nu\left(\theta - \theta_f\right).$$

Define

$$g(R) = \sqrt{\frac{R^2}{\nu^2}-1} + \sin^{-1}\left(\frac{\nu}{R}\right), \tag{21}$$

$$\implies \nu\left(g(R) - g(R_f)\right) = \nu\left(\theta - \theta_f\right)$$
$$\implies \theta\left(R; R_f, \theta_f\right) = g(R) - g(R_f) + \theta_f, \quad \theta_f \le \theta < \pi. \tag{22}$$

Setting $\theta_f = 0$ in (22) (i.e., $\theta(R; R_f, 0)$) describes the equilibrium flow field for the *Game of Distance* (i.e., assuming the Defender can drive $\theta \to 0$ before the Attacker can reach the target). The curve in (22) is the involute of a circle of radius ν.

Up until now, we have considered θ to be in the range $[0, \pi)$, however, the results apply to the range $(-\pi, 0]$ with some slight modification.

Lemma 1. The surface

$$\mathscr{D} \equiv \{\mathbf{x} \mid \theta = \pi\}, \tag{23}$$

is a Dispersal Surface (c.f. [1]) wherein the Defender can choose either $u_D = 1$ or $u_D = -1$ and both choices are optimal. Furthermore, when $\theta < 0$, the equilibrium controls are given by $u_D^* = -1$ and $\sin\psi^* < 0$.

Proof. By definition, points on a Dispersal Surface have two or more associated equilibrium trajectories which yield the same Value. We will show that (23) is indeed a Dispersal Surface by constructing a pair of equilibrium trajectories that integrate back to the same point on $\mathscr{D}$. Consider an initial state on the Dispersal Surface, $\mathbf{x}_{\mathscr{D}} = (R_0, \pi) \in \mathscr{D}$. The system (19) describes the evolution of R and θ in backwards time assuming $\eta > 0$. Now, let $\mathbf{x}_f \equiv (R_f, \theta_f)$ (where $R_f > 1$, $\theta_f \ge 0$) be the terminal state, which, when integrated through the retrograde kinematics (19), yields the initial point $\mathbf{x}_{\mathscr{D}}$. A symmetric solution can be constructed by switching the sign of η and θ_f, then integrating the retrograde kinematics back to $\mathbf{x}_{\mathscr{D}}$. Now, let $\eta < 0$; then $\sin\psi^* < 0$ from (15), and $u_D^* = -1$ from (16). Substitution into the Hamiltonian at final time yields $\eta = -\nu R_f \sqrt{\frac{1}{R_f^2 - \nu^2}}$. Substituting all of these into the Hamiltonian at general time yields the same expression for σ_R as in (18). Then, from (2), the retrograde kinematics are

$$\mathring{R} = \nu\sqrt{1 - \frac{\nu^2}{R^2}}, \qquad \mathring{\theta} = \frac{\nu^2}{R^2} - 1.$$

Clearly, these are the same kinematics as in (19) except the sign of $\mathring{\theta}$ is reversed. These kinematics can be integrated back from the symmetric terminal point $(R_f, -\theta_f)$ to the point $(R_0, -\pi)$, which is equivalent to $\mathbf{x}_{\mathscr{D}}$. This pair of trajectories emanating (forward in time) from $\mathbf{x}_{\mathscr{D}}$ have the same Value for all terminal cost functionals of R_f and $|\theta_f|$. Note this method for proving the presence of a Dispersal Surface is similar to the one used for a problem with similar dynamics in [21]. □

As a consequence, Assumption 2 may be relaxed, and the state space may be expanded to $\theta \in [-\pi, \pi]$.

Theorem 1 (*Game of Distance* Solution). The equilibrium state feedback control strategies for the *Game of Distance* are given by

$$\psi^* = \mathrm{sign}(\theta)\sin^{-1}\left(\frac{\nu}{R}\right), \qquad u_D^* = \mathrm{sign}(\theta). \tag{24}$$

The Value of the game is

$$V_d(R, \theta) = -R_f = -g^{-1}\left(g(R) - |\theta|\right). \tag{25}$$

Proof. The expression for ψ^* is obtained by substituting (17) and (18) into (15), taking into account the sign of θ (due to Lemma 1). Similarly, the Defender strategy is given by (16), accounting for Lemma 1. The corresponding form of (22) for the *Game of Distance* is

$$\theta\left(R; R_f\right) = g(R) - g(R_f). \tag{26}$$

Thus, (25) is obtained by rearranging this expression and solving for R_f, with $g(\cdot)$ defined as in (21). Because V_d is defined using the inverse of the function g, it is necessary to show that $g(R)$ is monotonic. Taking the derivative of (21) w.r.t. R gives

$$\frac{\mathrm{d}g}{\mathrm{d}R} = \frac{\sqrt{R^2 - \nu^2}}{\nu R}.$$

From Assumption 1, it must be that $0 < \nu < 1$, and from Assumption 3 it must be that $R > 1$ throughout the game. So we have $R > \nu$ and $R,\ \nu > 0$, which implies that $g(R)$ is monotonic. □

The Value function does not have a closed form analytic expression since g^{-1} cannot be expressed in closed form.

The limiting case for the *Game of Distance* is one in which $R_f \to 1$; thus the surface

$$\theta_{GoK}(R) = g(R) - g(1) \tag{27}$$

partitions the state space into regions of win for the Defender and Attacker, respectively,

$$\mathscr{R}_D = \{\mathbf{x} \mid |\theta| \le \theta_{GoK}(R)\} \tag{28}$$
$$\mathscr{R}_A = \{\mathbf{x} \mid |\theta| > \theta_{GoK}(R)\}. \tag{29}$$

Note that the value domain of $g(R)$ is $[g(1), \infty)$ since $R \ge 1$ and g is monotonic; from (28) $|\theta| \le \theta_{GoK}(R)$ in $\mathscr{R}_D$, so the argument to g^{-1} in (25) is $g(R) - |\theta| \ge g(R) - (g(R) - g(1)) = g(1)$ which is in the value domain of $g(R)$.

B. Attacker Win Scenario

In the region of the state space in which the Attacker 'wins' (i.e., can reach $R = 1$ while avoiding $\theta = 0$), we consider a *Game of Degree* wherein the players max/min the terminal separation angle; we refer to this as the *Game of Angle*. The cost/payoff functional is given as

$$J = \Phi\left(\mathbf{x}_f, t_f\right) = \theta_f. \tag{30}$$

The Attacker seeks to maximize the terminal separation angle whereas the Defender seeks to minimize. Termination occurs when the Attacker penetrates the target circle,

$$\phi\left(\mathbf{x}_f, t_f\right) = R_f - 1 = 0. \tag{31}$$

Theorem 2 (*Game of Angle* Solution)*.* The equilibrium state feedback strategies for the *Game of Angle* match those of the *Game of Distance*, i.e., are given by (24). The Value function is given by

$$V(R,\theta) = \theta_f = \theta - g\left(R\right) + g\left(1\right). \tag{32}$$

Proof. This proof is based upon showing satisfaction of the sufficient condition for equilibrium via substitution of the proposed equilibrium strategies and Value function into the Hamilton-Jacobi-Isaacs (HJI) equation [1],

$$\min_{u_D} \max_{\psi} \left\{ l\left(\mathbf{x}, u_D, \psi, t\right) + \frac{\partial V}{\partial t} + V_{\mathbf{x}} \cdot f\left(\mathbf{x}, u_D, \psi, t\right) \right\} = 0, \tag{33}$$

where $V_{\mathbf{x}}$ is the vector $\begin{bmatrix} \frac{\partial V}{\partial R} & \frac{\partial V}{\partial \theta} & \frac{\partial V}{\partial \beta} \end{bmatrix}^\top$, and l represents an integral cost component. First, note that the cost, (30), has no integral component, and thus $l = 0$. Also, the proposed Value function, (32) is not an explicit function of time and thus $\frac{\partial V}{\partial t} = 0$. The vector $V_{\mathbf{x}}$ is obtained by differentiating (32) w.r.t. each state,

$$V_{\mathbf{x}} = \begin{bmatrix} \frac{-\sqrt{R^2 - \nu^2}}{R\nu} & 1 & 0 \end{bmatrix}.$$

The (forward) equilibrium dynamics, f, are given by the negative of (19). Substituting all of these expressions into (33) gives

$$\frac{\partial V}{\partial R}\dot{R} + \frac{\partial V}{\partial \theta}\dot{\theta} = \left(\frac{-\sqrt{R^2 - \nu^2}}{R\nu}\right)\left(-\nu\sqrt{1 - \frac{\nu^2}{R^2}}\right) + \left(\frac{\nu^2}{R^2} - 1\right) = 0.$$

The proposed Value function is continuous and continuously differentiable (except on the Dispersal Surface, $\mathscr{D}$), and it satisfies the HJI hyperbolic PDE. □

Remark 1*.* Theorem 1 (as well as Theorems 3 and 4) can be verified in a similar fashion (i.e., by substituting the respective Value functions into the HJI to show it is satisfied). The analysis would be nearly identical to the above proof and is therefore omitted.

C. Full Equilibrium Flow Field

With the analysis in Sections II-A and II-B, the entire (usable) state space can be filled with equilibrium trajectories. Figure 2 shows (22) and (26) in the Attacker win and lose regions, respectively.

Lemma 2*.* The Attacker's equilibrium trajectory is a straight line in the inertial (non-rotating) (x, y)-plane.

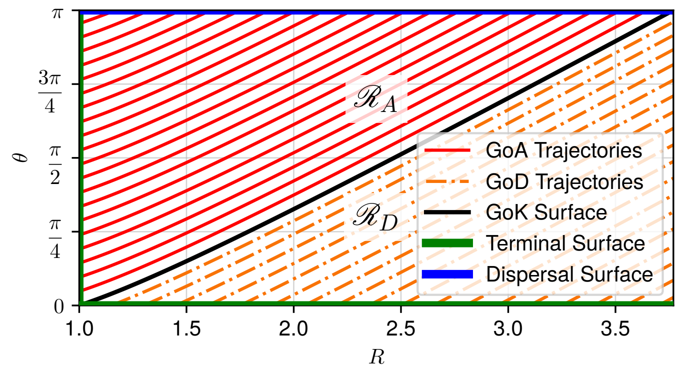

Fig. 2. Full equilibrium flow field with $\nu = 0.8$

Proof. Consider Fig. 1 which shows the Attacker's heading angle, $\tilde{\psi}$, w.r.t. the inertial (x, y)-plane. The following relation holds

$$\tilde{\psi} = \beta + \theta + \pi - \psi$$

Thus, the time derivative of the global Attacker heading angle is given as

$$\dot{\tilde{\psi}} = \dot{\beta} + \dot{\theta} - \dot{\psi}$$

Substituting (24) and (19) into the above gives

$$\begin{aligned} \dot{\tilde{\psi}} &= 1 + \frac{\nu^2}{R^2} - 1 - \frac{\partial}{\partial t}\sin^{-1}\left(\frac{\nu}{R}\right) \\ &= \frac{\nu^2}{R^2} - \left(\frac{-1}{\sqrt{1 - \frac{\nu^2}{R^2}}}\right)\left(\frac{\nu}{R^2}\right)\dot{R} \\ &= \frac{\nu^2}{R^2} + \left(\frac{1}{\sqrt{1 - \frac{\nu^2}{R^2}}}\right)\left(\frac{\nu}{R^2}\right)\left(-\nu\sqrt{1 - \frac{\nu^2}{R^2}}\right) \\ &= 0. \end{aligned}$$

Because $\dot{\tilde{\psi}} = 0$, the global Attacker heading angle is constant, and thus the Attacker's path is a straight-line in the inertial (x, y)-plane. □

D. Alternative Attacker Win Scenario

Depending on the particular physical application or interpretation of the scenario, the Attacker may be interested in penetrating the target circle in minimum time. For example, if the Attacker is a munition of some kind, it may not matter how far or close the Defender is at the time of penetration. Formulating a game of $\min / \max$ terminal angular separation, on the other hand, may make sense when the Attacker is some kind of vehicle or person who seeks to intrude inside the target circle while avoiding, as much as possible, coming into contact with the Defender.

The cost functional considered here is

$$J = \Phi\left(\mathbf{x}_f, t_f\right) = -t_f. \tag{34}$$

Once again, the negative sign is used to adhere to the convention established in previous sections in which the Attacker and Defender are the maximizer and minimizer, respectively. Since penetration is assumed to occur, the termination condition is given by (31), i.e., when the Attacker reaches the target, and it must be ensured that $\theta(t) \neq 0$ *en route*. We refer to this scenario as the *Game of Min Time*.

Proposition 1. *For the zero-sum differential game whose cost is given by* (34)*, the Defender's equilibrium control is*

$$u_D^* = \operatorname{sign}(\theta). \tag{35}$$

Proof. First, note the proposed Defender control is the same as in both of the *Games of Degree* considered previously (c.f. Theorems 1 and 2). The Defender's essential goal in all of these scenarios is to align with the Attacker, if possible, or otherwise impede the Attacker somehow. Likewise, the Attacker must avoid alignment with the Defender in order to achieve its objectives. Thus the Defender is always interested in driving $\theta \to 0$ as quickly as possible. From (2), the fastest way to achieve this is traversing, at maximum speed, in the direction of the Attacker, i.e., by implementing (35). □

Lemma 3. For any scenario in which the Defender implements $u_D = \operatorname{sign}(\theta)$, it is necessary for $\mathbf{x} \in \mathscr{R}_A$ for all $t \in [0, t_f]$ in order for the Attacker to achieve penetration, i.e., $R_f = 1$ with $\theta > 0$ for all $t \in [0, t_f]$.

Proof. This Defender control is the equilibrium control for the *Game of Distance*. In the *Game of Distance*, which is played when $\mathbf{x} \notin \mathscr{R}_A$, the Attacker seeks to come as close to the target circle before alignment with the Defender occurs. The limiting case is when the Attacker reaches the target circle at the exact moment alignment occurs. If it had gotten there sooner, then it must have been the case that $\mathbf{x} \in \mathscr{R}_A$ since the equilibrium controls are the same as in the *Game of Angle*. In general, $R_f > 1$, which means that the Defender achieves alignment before the Attacker achieves penetration *under the best possible Attacker control*. □

Lemma 3 expresses the necessary condition for the Attacker to penetrate the target circle, which is applicable to any scenario in which the Defender seeks to align with the Attacker. Thus the *Game of Min Time* takes place in $\mathscr{R}_A$. Let us focus on the case where $\theta \in [0, \pi]$. The terminal condition is

$$\phi(\mathbf{x}_f, t_f) = R_f - 1 = 0, \ \theta \in [0, \pi]. \tag{36}$$

Thus the terminal surface is the zero-level set of ϕ and is left-discontinuous at $\theta = 0$. Additionally, the boundary of the terminal surface, $(R, \theta) = (1, 0)$, lies on the boundary of the state space, $\partial\mathscr{R}_A$.

The Hamiltonian is

$$\mathscr{H} = -\lambda_R \nu \cos\psi + \lambda_\theta \left(\frac{\nu}{R}\sin\psi - 1\right) + \lambda_\beta, \tag{37}$$

where $\lambda \equiv \begin{bmatrix}\lambda_R & \lambda_\theta & \lambda_\beta\end{bmatrix}$. Both λ_θ and λ_β are constant since $\frac{\partial \mathscr{H}}{\partial \theta} = 0$ and $\frac{\partial \mathscr{H}}{\partial \beta} = 0$, respectively. In general, the transversality condition [20] is

$$\lambda^\top(t_f) = \frac{\partial \Phi}{\partial \mathbf{x}_f} + \mu \frac{\partial \phi}{\partial \mathbf{x}_f}. \tag{38}$$

The term $\frac{\partial \phi}{\partial \theta}$ is well-defined when $\theta \in (0, \pi)$, but it is undefined at the "corner point" $(R, \theta) = (1, 0)$. We first treat the former case.

E. General Case

Lemma 4. For the zero-sum differential game whose cost is given by (34), the equilibrium Attacker control is

$$\psi^* = 0, \qquad \forall \mathbf{x} \text{ s.t. } \theta > \frac{R-1}{\nu}. \tag{39}$$

Proof. Specializing (38) gives

$$\lambda^\top(t_f) = 0 + \mu \begin{bmatrix}1 & 0 & 0\end{bmatrix}, \tag{40}$$

thus $\lambda_\theta, \ \lambda_\beta = 0$ for all $t \in [0, t_f]$. Note that we have assumed that $\theta_f > 0$ which implies that $\frac{\partial \phi}{\partial \theta_f}$ exists and equals 0. Substituting into the Hamiltonian, (37), gives

$$\mathscr{H} = -\lambda_R \nu \cos\psi,$$

which is maximized for $\cos\psi^* = -\operatorname{sign}(\lambda_R) = \pm 1$. It is obvious that the Attacker must run *toward* the target circle, hence (39) holds.

Now, we must ensure that the assumption $\theta_f > 0$ is valid. Under the equilibrium control strategies $u_D^* = \operatorname{sign}(\theta)$ and $\psi^* = 0$ the R and θ dynamics are

$$\dot{R} = -\nu, \qquad \dot{\theta} = -1, \tag{41}$$

and thus $\frac{\mathrm{d}\,\theta}{\mathrm{d}R} = \frac{1}{\nu}$, implying that the trajectories are straight lines in the (R, θ) plane. Furthermore, the unconstrained equilibrium flowfield is

$$\theta(R) = \frac{1}{\nu}(R - R_0) + \theta_0. \tag{42}$$

The critical case occurs when the constraint activates at the precise moment that the Attacker reaches the target (i.e., $\theta_f = 0$). The time to traverse from their initial positions must be equal, giving

$$\theta_c(R) = \frac{R-1}{\nu}. \tag{43}$$

If $\theta > \theta_c$, then $\theta_f > 0$ under equilibrium play, hence the specification in (39). □

However, if $\theta < \theta_c$ and the Attacker aims at the circle center, the Defender can drive $\theta \to 0$ before the Attacker reaches the target.

F. Corner Case

Now we treat the case wherein the game terminates on "corner point" $(R, \theta) = (1, 0)$. Recall the fact that $\frac{\partial \phi}{\partial \theta_f}$ is undefined when $\theta_f = 0$ which results in λ_{θ_f} being free. The consequence is that the incoming equilibrium trajectory to the corner point is not unique, unlike elsewhere in the terminal surface. Therefore, a family of trajectories, beginning from a range of initial conditions all terminate at $(R, \theta) = (1, 0)$. A similar situation arises in pursuit-evasion scenarios with a finite capture radius [22]–[24] For notational convenience, let $\lambda_{R_f} \equiv \mu$ and $\lambda_{\theta_f} \equiv \eta$; $\lambda_{\beta_f} = 0$, as before.

Lemma 5. For the zero-sum differential game whose cost is given by (34) the equilibrium Attacker control is

$$\psi^* = \sin^{-1}\left(\frac{\kappa}{R}\right), \ \kappa \in [0, \nu], \qquad \forall \mathbf{x} \in \mathscr{R}_A \text{ s.t. } \theta \leq \frac{R-1}{\nu}, \tag{44}$$

where κ satisfies

$$0 = \frac{1}{\nu}\left(\sqrt{R^2-\kappa^2}-\sqrt{1-\kappa^2}\right)-\theta-\sin^{-1}\kappa+\sin^{-1}\left(\frac{\kappa}{R}\right). \tag{45}$$

The Attacker trajectory is a straight line terminating at $(R,\theta)=(1,0)$.

Proof. The Hamiltonian, (37), evaluated at final time is

$$\mathscr{H}_f = -\mu\nu\cos\psi_f + \eta\left(\nu\sin\psi_f - 1\right). \tag{46}$$

The equilibrium Attacker heading must maximize $\mathscr{H}_f$, thus

$$\cos\psi_f^* = \frac{-\mu}{\sqrt{\mu^2+\eta^2}}, \qquad \sin\psi_f^* = \frac{\eta}{\sqrt{\mu^2+\eta^2}}. \tag{47}$$

Substituting this terminal Attacker heading back into (46) gives

$$\mathscr{H}_f = \nu\sqrt{\mu^2+\eta^2}-\eta. \tag{48}$$

The terminal Hamiltonian value is also specified by [20]

$$\mathscr{H}_f = -\frac{\partial\Phi}{\partial t_f}-\mu\frac{\partial\phi}{\partial t_f} = -\left(-1\right)-0=1 \tag{49}$$

Substituting back into (48) and rearranging gives

$$\sin\psi_f^* = \frac{\nu\eta}{1+\eta}. \tag{50}$$

Define, for convenience, $\kappa \equiv \sin\psi_f$. The value of κ is bounded,

$$\kappa\in\left[0,\nu\right]. \tag{51}$$

The lower bound is due to the fact that A ought not aim *towards* D; the upper bound is due to the fact that $\kappa>\nu$ would immediately push the state of the system out of $\mathscr{R}_A$.

Because the system is time-autonomous, the value of the Hamiltonian is constant, i.e., $\mathscr{H}(t)=1$ for all $t\in[0,t_f]$. Rewriting (46) and (47) at general time and solving for λ_R^2 gives

$$\lambda_R^2 = \left(\frac{1+\eta}{\nu}\right)^2-\frac{\eta^2}{R^2}. \tag{52}$$

Similarly, the Attacker heading at general time is

$$\begin{aligned}\sin\psi &= \frac{\eta}{R\sqrt{\lambda_R^2+\frac{\eta^2}{R^2}}}\\ &= \frac{\nu\eta}{R\left(1+\eta\right)} = \frac{\sin\psi_f}{R} = \frac{\kappa}{R}\end{aligned} \tag{53}$$

Therefore, Lemma 2 holds here as well since κ is a constant; that is, the Attacker's path is a straight line in the non-rotating (x,y)-frame.

Concerning the determination of κ for a general position $R>1$ and $\theta\in[\theta_{GoK},\theta_c]$, we turn to geometry. Let the point I be the point on the target circle in which A will terminate; by construction, this must be the point D terminates as well. A right triangle is formed by $\triangle ATO$ where T is the tangent point of the extension of A's trajectory on a circle of radius κ, and O is the target circle's center. The hypotenuse of $\triangle ATO$ is R. Another right triangle is formed by $\triangle ITO$; its hypotenuse is 1. See Fig. 3 for a representation of the geometry. The distance traveled by the Attacker is

$$\begin{aligned}\overline{AI} &= \overline{AT}-\overline{TI}\\ &= \sqrt{R^2-\kappa^2}-\sqrt{1-\kappa^2}.\end{aligned} \tag{54}$$

The Defender must cover an angular distance θ as well as the circular sector between A and I. Define $\rho\equiv\angle AOI$, which is given by

$$\rho = \sin^{-1}\kappa-\sin^{-1}\left(\frac{\kappa}{R}\right). \tag{55}$$

Define m as the difference in Attacker and Defender travel times to the point I:

$$m\left(\kappa\right) = \frac{1}{\nu}\overline{AI}-\left(\theta+\rho\right). \tag{56}$$

Then (45) is obtained by substituting in (54) and (55) and setting equal to zero, which represents simultaneous arrival to the point I. The value of κ for which this occurs may be obtained numerically. □

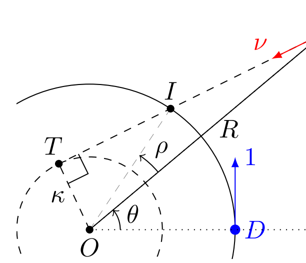

Fig. 3. Schematic of the *Game of Min Time* scenario in which A takes an "evasive" path in order to arrive at the point I simultaneously with D.

In the *Game of Min Time* nothing is gained by increasing θ_f; the Attacker either heads directly towards the Target without regard for the Defender or performs the minimum "evasion" necessary to reach the Target, whereupon $\theta_f=0$. The equilibrium kinematics may obtained by substituting (44) into (2). A closed form expression of the equilibrium trajectory through the (R,θ) space is obtained via a process similar to (19)–(22) in Section II-A2:

$$\theta(R;\kappa) = \frac{\sqrt{R^2-\kappa^2}}{\nu}+\sin^{-1}\frac{\kappa}{R}-\frac{\sqrt{1-\kappa^2}}{\nu}-\sin^{-1}\kappa \tag{57}$$

The full solution of the *Game of Min Time* is depicted in Fig. 4.

III. Two Defenders

In this section, we consider the circular target guarding game with two Defenders, D_1 and D_2, with the following assumption:

Assumption 4. The two Defenders share the same maximum speed: $v_{D_{1_{\max}}}=v_{D_{2_{\max}}}=v_{D_{\max}}$.

The scenario is depicted in Fig. 5, and the (nondimensional) kinematics of the system are given as

$$f\left(\mathbf{x},\mathbf{u},t\right)=\dot{\mathbf{x}}=\begin{bmatrix}\dot R\\ \dot\gamma\\ \dot\alpha\\ \dot\beta\end{bmatrix}=\begin{bmatrix}-\nu\cos\psi\\ \frac{\nu}{R}\sin\psi-\frac{1}{2}\left(u_{D_1}+u_{D_2}\right)\\ \frac{1}{2}\left(u_{D_1}-u_{D_2}\right)\\ u_{D_1}\end{bmatrix}. \tag{58}$$

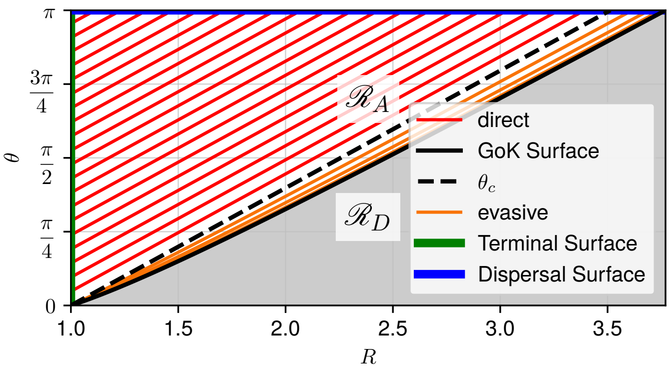

Fig. 4. Equilibrium flowfield for the *Game of Min Time* with $\nu = 0.8$. The "direct" trajectories are straight lines with slope $\frac{1}{\nu}$. The "evasive" trajectories are described by (57) with various κ. The dashed black line is the critical trajectory described by (43).

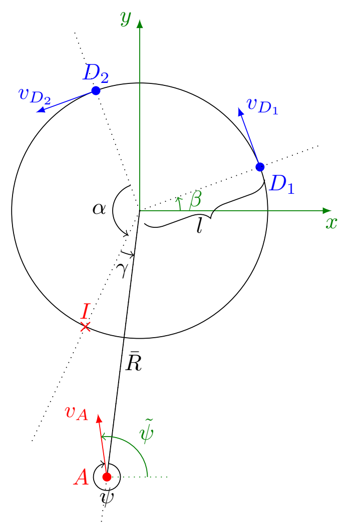

Fig. 5. Circular perimeter patrol with two Defenders and one Attacker.

The angle α is measured from D_2 to the angular bisector (on the side of A) of the positions of D_1 and D_2. Similarly, the angle γ is measured as A's angular offset w.r.t. this bisector.

Assumption 5. The relative angular position of the Attacker is bounded such that $-\alpha \leq \gamma \leq \alpha$.

Although we impose Assumption 5, it is of little consequence since the forthcoming solution would still apply for γ outside this range by, for example, switching the designation of D_1 and D_2. Just as in the analysis of the one-on-one game, there are three "games" or questions of interest: 1) can the Attacker reach the target (the *Game of Kind*), 2) what is the equilibrium terminal angular separation between the Attacker and the closest Defender (the *Game of Angle*), and 3) what is the equilibrium terminal distance from the target center (the *Game of Distance*).

Note that the rotation of the system w.r.t. the global x-axis, β, has no effect on the optimality of the trajectories as in the one-on-one analysis and is therefore omitted in the following.

A. *Game of Degree When Attacker Wins*

Here, we consider the *Game of Angle* which applies to the scenario when the Attacker is able to reach the target ($R_f = 1$). The cost functional is given as

$$J = \Phi(\mathbf{x}_f, t_f) = \alpha_f - |\gamma_f|, \tag{59}$$

and we seek the Value of the game:

$$V(\mathbf{x}) = \min_{\mathbf{u}(\cdot)} \max_{\psi(\cdot)} J = \max_{\psi(\cdot)} \min_{\mathbf{u}(\cdot)} J. \tag{60}$$

This game terminates when the following condition is satisfied

$$\phi(\mathbf{x}_f, t_f) = R_f - 1 = 0. \tag{61}$$

1) First Order Necessary Conditions for Optimality: First, form the Hamiltonian as

$$\begin{aligned} \mathscr{H} = -\lambda_R \nu \cos\psi + \lambda_\gamma \left(\frac{\nu}{R} \sin\psi - \frac{1}{2}\left(u_{D_1} + u_{D_2}\right)\right) \\ + \lambda_\alpha \frac{1}{2}\left(u_{D_1} - u_{D_2}\right). \end{aligned} \tag{62}$$

The equilibrium adjoint dynamics obey [20]

$$\dot{\lambda}_R = -\frac{\partial \mathscr{H}}{\partial R} = \lambda_\gamma \frac{\nu}{R^2} \sin\psi, \tag{63}$$

$$\dot{\lambda}_\gamma = -\frac{\partial \mathscr{H}}{\partial \gamma} = 0, \tag{64}$$

$$\dot{\lambda}_\alpha = -\frac{\partial \mathscr{H}}{\partial \alpha} = 0. \tag{65}$$

From the transversality condition [20], the equilibrium terminal adjoint values satisfy

$$\begin{aligned} \lambda^\top(t_f) &= \frac{\partial \Phi}{\partial \mathbf{x}_f} + \mu \frac{\partial \phi}{\partial \mathbf{x}_f} \\ \implies \quad \lambda_R(t_f) &= \mu, \\ \lambda_\gamma(t_f) &= -\operatorname{sign}(\gamma_f), \\ \lambda_\alpha(t_f) &= 1. \end{aligned} \tag{66}$$

Because $\dot{\lambda}_\gamma = \dot{\lambda}_\alpha = 0$ we have $\lambda_\gamma(t) = -\operatorname{sign}(\gamma_f)$ and $\lambda_\alpha(t) = 1$ for all $t \in [t_0, t_f]$.

2) Solution Characteristics:

Lemma 6. For games terminating with $\gamma_f \neq 0$, the game's Value function and optimal strategies are that of the one-on-one game: $V = \alpha - |\gamma| - g(R) + g(1)$, and $\psi^* = \sin^{-1}\left(-\frac{\nu}{R}\operatorname{sign}(\gamma_f)\right)$. The second Defender is redundant.

Proof. Suppose that $\gamma_f < 0$; substituting the corresponding λ_γ and λ_α values into (62) gives

$$\mathscr{H} = -\lambda_R \nu \cos\psi - \frac{\nu}{R}\sin\psi - u_{D_2}. \tag{67}$$

Note that u_{D_1} does not appear in (67) and thus D_1 has no effect on the optimality of the trajectory and is therefore redundant. Again, since the final time is free, the Hamiltonian, at terminal time, is subject to (14) [20]; that is, $\mathscr{H}(t_f) = 0$. Since (58) are autonomous, we have $\mathscr{H}(t) = 0$ for all $t \in [t_0, t_f]$. Therefore, (67) is identical to the Hamiltonian for the one-on-one case between the Attacker and D_2. Furthermore, the terminal condition is the same, and the cost functional is identical since $\theta = \alpha - |\gamma| = J$, in this case. Thus, the Value function for the one-on-one case, (32), and the equilibrium

Attacker heading control, (24) are the solution for this game (making the appropriate substitution of $\theta = \alpha - |\gamma|$). The $-\operatorname{sign}$ in the ψ^* expression, in this case, accounts for the case when $\gamma_f > 0$ in which the game plays out between the Attacker and D_1, by symmetry. In that case, the scenario is a mirror image of Fig. 1 and the sign of u_{D_1} is reversed (i.e., D_1 moves clockwise) as is the sign of $\sin\psi^*$. □

Since $\gamma_f \neq 0$ corresponds to either one-on-one game, we focus our attention on the case when $\gamma_f = 0$. When $\gamma_f = 0$, the Attacker terminates at a position which is equidistant from the two defenders. Note that, according to (66), $\lambda_\gamma(t_f) = \lambda_{\gamma_f}$ is undefined when $\gamma_f = 0$. As before, the Defenders seek to minimize the Hamiltonian, (62):

$$\begin{aligned} u^*_{D_1}, u^*_{D_2} &= \underset{u_{D_1}, u_{D_2}}{\arg\min} \mathscr{H} \\ &= \underset{u_{D_1}, u_{D_2}}{\arg\min} u_{D_1}(1 - \lambda_\gamma) + u_{D_2}(-1 - \lambda_\gamma). \end{aligned} \tag{68}$$

Now, according to (68), if $\lambda_\gamma > 1$ or $\lambda_\gamma < -1$ then $u^*_{D_1} = u^*_{D_2}$ which means the Defenders should move in the same direction. However, if this were the case then $\dot{\alpha} = 0$ which is clearly undesirable since α appears in the cost, J. Thus the value of λ_γ is bounded:

$$-1 \leq \lambda_\gamma \leq 1. \tag{69}$$

By inspection, it is clear that the Defenders should seek to minimize $\dot{\alpha}$ which occurs for

$$u^*_{D_1} = -1, \quad u^*_{D_2} = 1. \tag{70}$$

substituting in (66) and (70) into (62) leads to an expression for λ_R:

$$\begin{aligned} \mathscr{H}(t) = 0 &= \nu\sqrt{\lambda_R^2 + \frac{\lambda_\gamma^2}{R^2}} - 1 \\ \implies \lambda_R &= \pm\sqrt{\frac{1}{\nu^2} - \frac{\lambda_\gamma^2}{R^2}}. \end{aligned}$$

Since $\dot{R}_f \propto \cos\psi_f \propto \nu$ it must be that $\lambda_{R_f}, \nu < 0$ in order for the state of the system to penetrate the boundary. In order to maximize the Hamiltonian, it must be that $\sin\psi^* \propto \lambda_\gamma$; thus, from (63), $\dot{\lambda}_R(t) < 0$ for all $t \in [t_0, t_f]$. Therefore, $\lambda_R(t) < 0$ for all $t \in [t_0, t_f]$, which leads to

$$\lambda_R = -\sqrt{\frac{1}{\nu^2} - \frac{\lambda_\gamma^2}{R^2}}. \tag{71}$$

Lemma 7. For games terminating with $\gamma_f = 0$, the equilibrium heading angle is

$$\psi^* = \sin^{-1}\left(\lambda_\gamma \frac{\nu}{R}\right), \tag{72}$$

and is bounded by $-\sin^{-1}\left(\frac{\nu}{R}\right) \leq \psi^* \leq \sin^{-1}\left(\frac{\nu}{R}\right)$.

Proof. Substituting (71) with (70) into (62) gives

$$\mathscr{H} = 0 = -\nu\sqrt{\frac{1}{\nu^2} - \frac{\lambda_\gamma^2}{R^2}}\cos\psi + \lambda_\gamma \frac{\nu}{R}\sin\psi. \tag{73}$$

The Attacker seeks to maximize the Hamiltonian, and thus

$$\cos\psi^* = -\sqrt{1 - \frac{\nu^2\lambda_\gamma^2}{R^2}}, \quad \sin\psi^* = \lambda_\gamma \frac{\nu}{R}, \tag{74}$$

and $-1 \leq \lambda_\gamma \leq 1$ according to (69), hence $-\frac{\nu}{R} \leq \sin\psi^* \leq \frac{\nu}{R}$. □

Lemma 8. The trajectories corresponding to $\lambda_\gamma = \pm 1$ separate the state space into regions of asymmetric termination ($\gamma_f \neq 0$) and symmetric termination ($\gamma_f = 0$).

Proof. Suppose $\lambda_\gamma = 1$, then the Attacker's equilibrium strategy is identical to the one-on-one game with D_2 (c.f. (24)). The trajectory is a straight line in the global (x, y)-frame since the one-on-one game Attacker trajectories are straight (due to Lemma 2). Trajectories with $\lambda_\gamma < 1$ lie on one side of this surface and one-on-one trajectories (against D_2) lie on the other side. □

Lemma 9. Attacker trajectories resulting in symmetric termination ($\gamma_f = 0$) are straight lines in the (x, y)-plane terminating at a point I, where

$$I = \begin{bmatrix} I_x \\ I_y \end{bmatrix} = \begin{bmatrix} \cos(\beta_0 - \alpha_0) \\ \sin(\beta_0 - \alpha_0) \end{bmatrix}. \tag{75}$$

Proof. Just as in Lemma 2, the Attacker trajectory is shown to be a straight line in the inertial frame via direct substitution of the equilibrium strategies. Consider Fig. 5 which shows the Attacker's heading angle $\tilde{\psi}$, w.r.t. the inertial (x, y)-plane. It is expressed

$$\tilde{\psi} = \beta + (2\pi - 2\alpha) + \alpha + \gamma - \psi,$$

and its time derivative is

$$\dot{\tilde{\psi}} = \dot{\beta} - \dot{\alpha} + \dot{\gamma} - \dot{\psi}.$$

Substitution of the kinematics, (58), and the equilibrium controls, (70) and (72), gives

$$\begin{aligned} \dot{\tilde{\psi}} &= u_{D_1} - \frac{1}{2}(u_{D_1} - u_{D_2}) + \frac{\nu}{R}\sin\psi - \\ &\quad \frac{1}{2}(u_{D_1} + u_{D_2}) - \frac{\partial}{\partial t}\sin^{-1}\lambda_\gamma \frac{\nu}{R} \\ &= -1 + 1 + \frac{\lambda_\gamma \nu^2}{R^2} - 0 - \left(\frac{-\lambda_\gamma \nu}{R^2\sqrt{1 - \frac{\lambda_\gamma^2\nu^2}{R^2}}}\right)(-\nu\cos\psi) \\ &= \frac{\lambda_\gamma\nu^2}{R^2} - \frac{\lambda_\gamma\nu^2}{R^2} = 0 \end{aligned}$$

Since the Attacker heading in the inertial (x, y)-plane is constant, the Attacker path is a straight line. For symmetric termination, the state of the system lies at $R = 1$ and $\gamma = 0$. The $\gamma = 0$ angle corresponds to $\beta - \alpha$. Because $u^*_{D_1} = -1$ and $u^*_{D_2} = 1$ (due to (74)) we have $\dot{\alpha} = -1 = \dot{\beta}$ and thus the angle $\beta - \alpha$ is invariant in the global (x, y)-plane. □

Lemma 10. For symmetric termination ($\gamma_f = 0$), the separating surface of the *Game of Kind* in the global (x, y)-plane is given by a circular arc centered I with radius $\nu\alpha_0$ whose bounds are defined by $\sin^{-1}(-\nu)$ and $\sin^{-1}(\nu)$ relative to the $\gamma = 0$ axis.

Proof. Symmetric termination trajectories terminate at I, defined and according to Lemma 9. The limiting case occurs when the Attacker reaches the target circle at the exact moment in which the Defenders reach I (i.e. $\alpha_f \to 0$). Due to (70), we have $\dot{\alpha} = -1$. Therefore, the Defenders reach $\alpha = 0$ in α_0 time. Symmetric termination trajectories may thus extend from I for a maximum distance of $\nu\alpha_0$; beyond this distance, the Attacker cannot reach the target. The Attacker trajectories are straight, also due to Lemma 9, thus the *Game of Kind* surface is a circular arc. The bounds of the circular arc are given directly by the range of ψ_f^* which is obtained by substituting $R = 1$ into (72) and applying the bounds stated in Lemma 7. □

The regions $\mathscr{R}_{A_1}$ and $\mathscr{R}_{A_2}$ are the sets of states for which the game terminates with $\gamma_f > 0$ (one-on-one with D_1) and $\gamma_f < 0$ (one-on-one with D_2), respectively (c.f. Lemma 6). Similarly, the region $\mathscr{R}_{A_{1,2}}$ is the set of states for which the game terminates with $\gamma_f = 0$ and is completely specified by Lemmas 8–10. The polar distance at which the *Game of Kind* surface switches from the one-on-one surface, governed by (27), and the two-on-one surface, described in Lemma 10, is given by

$$R_s = +\sqrt{\nu^2\alpha^2 + 1 + 2\nu\alpha\sqrt{1-\nu^2}}, \tag{76}$$

which is derived from the Law of Cosines (see Fig. 7).

Theorem 3. In the region $\mathscr{R}_{A_{1,2}}$, the equilibrium Attacker heading angle is given by

$$\psi^* = \sin^{-1}\left(\frac{\sin\gamma}{p}\right), \tag{77}$$

and the associated Value function is

$$V(\mathbf{x}) = \alpha_f = \alpha - \frac{p}{\nu}, \tag{78}$$

where

$$p = +\sqrt{R^2 + 1 - 2R\cos\gamma}.$$

Proof. Consider the triangle formed by the Attacker's position, the target circle center, and the point I as defined in (75). By construction, the Attacker starts in $\mathscr{R}_{A_{1,2}}$ and its equilibrium trajectory must terminate at I due to Lemma 9. Let the distance traveled from A_0 to I be p, which can be obtained from the Law of Cosines (as defined above). Then, (77) can be obtained from the Law of Sines. The time taken to traverse this path is p/ν, and $\dot{\alpha} = -1$ (due to (70)), thus (78) follows. □

B. *Game of Degree When Attacker Loses*

In this section, we focus on the *Game of Distance* which applies to the scenario when A is not able to reach the target before one or both Defenders can align with A (i.e. $\alpha - |\gamma| = 0$). The cost functional is the same as in the one-on-one case, i.e., (3). This game terminates when the following condition is satisfied

$$\phi_d(\mathbf{x}_f, t_f) = \alpha_f - |\gamma_f| = 0. \tag{79}$$

1) First Order Necessary Conditions:

The Hamiltonian is

$$\mathscr{H}_d = -\sigma_R\nu\cos\psi + \sigma_\gamma\left(\frac{\nu}{R}\sin\psi - \frac{1}{2}(u_{D_1} + u_{D_2})\right) + \sigma_\alpha\frac{1}{2}(u_{D_1} - u_{D_2}), \tag{80}$$

and thus the equilibrium adjoint dynamics are

$$\dot{\sigma}_R = -\frac{\partial\mathscr{H}_d}{\partial R} = \sigma_\gamma\frac{\nu}{R^2}\sin\psi \tag{81}$$

$$\dot{\sigma}_\gamma = -\frac{\partial\mathscr{H}_d}{\partial\gamma} = 0 \tag{82}$$

$$\dot{\sigma}_\alpha = -\frac{\partial\mathscr{H}_d}{\partial\alpha} = 0. \tag{83}$$

From the transversality condition [20], the terminal adjoint values are

$$\sigma^\top(t_f) = \frac{\partial\Phi_d}{\partial\mathbf{x}_f} + \eta\frac{\partial\phi_d}{\partial\mathbf{x}_f} \tag{84}$$

$$= \begin{bmatrix}-1 & 0 & 0\end{bmatrix} + \eta\begin{bmatrix}0 & \pm 1 & 1\end{bmatrix}. \tag{85}$$

When $\gamma_f = 0$, however, the derivative $\frac{\partial\phi_d}{\partial\gamma_f}$, and thus σ_{γ_f}, is not defined. Evaluating (80) at final time and substituting in the terminal adjoint values gives

$$\mathscr{H}_d(t_f) = \nu\cos\psi_f + \sigma_{\gamma_f}\left(\frac{\nu}{R_f}\sin\psi_f - \frac{1}{2}(u_{D_1} + u_{D_2})\right) + \frac{\eta}{2}(u_{D_1} - u_{D_2}) \tag{86}$$

The Hamiltonian at terminal time is given by [20]

$$\mathscr{H}_d(t_f) = -\frac{\partial\Phi_d}{\partial t_f} - \eta\frac{\partial\phi_d}{\partial t_f} = 0. \tag{87}$$

2) Solution Characteristics:

Lemma 11. For games terminating with $\gamma_f \neq 0$, the game's Value function and optimal strategies correspond to the one-on-one game (c.f. Theorem 1). The second Defender is redundant.

Proof. The proof is similar to that of Lemma 6 in that the Hamiltonian is formed and a particular sign of γ_f is assumed, which results in reduction to the one-on-one Hamiltonian with identical cost and terminal boundary condition. If, for example, $\gamma_f < 0$ then $\sigma_{\gamma_f} = \eta$ and is σ_γ is constant since $\dot{\sigma}_\gamma = 0$. The Hamiltonian would be reduced to

$$\mathscr{H}_d = -\sigma_R\nu\cos\psi + \eta\left(\frac{\nu}{R}\sin\psi - u_{D_2}\right),$$

which matches exactly with the one-Attacker one-Defender Hamiltonian, (7). The cost functional (based on terminal distance, (3)) is the same and thus the two-Defender scenario reduces to the one-Defender scenario whenever $\gamma_f \neq 0$, by symmetry. □

Lemma 12. For games terminating with $\gamma_f = 0$, the equilibrium Attacker heading angle is

$$\psi^* = \sin^{-1}\left(\chi\frac{\nu}{R}\right), \qquad \chi \in [-1, 1]. \tag{88}$$

Proof. The proof is similar to that of Lemma 7, but with the associated first order necessary conditions for the *Game of*

Distance from the previous section. Now, if A and D_2 were to play the one-on-one *Game of Distance* D_2 would move counterclockwise, i.e., $u_{D_2} = 1$. The presence of D_1 ought not change the control of D_2 – counterclockwise is still the direction which closes the angular gap the between D_2 and A the fastest. Therefore, let $u^*_{D_{1_f}} = -1$ and $u^*_{D_{2_f}} = 1$. As before, the Defenders must minimize the Hamiltonian, including at final time. Thus from (87) we have

$$\begin{aligned} u^*_{D_{1_f}}, u^*_{D_{2_f}} &= \underset{u_{D_{1_f}}, u_{D_{2_f}}}{\arg\min}\ \mathscr{H} \\ &= \underset{u_{D_{1_f}}, u_{D_{2_f}}}{\arg\min}\ \left(-\sigma_{\gamma_f} + \eta\right) u_{D_{1_f}} + \\ &\qquad\qquad \left(-\sigma_{\gamma_f} - \eta\right) u_{D_{2_f}} \\ \implies u^*_{D_{1_f}} &= -\operatorname{sign}\left(-\sigma_{\gamma_f} + \eta\right) = -1, \\ u^*_{D_{2_f}} &= -\operatorname{sign}\left(-\sigma_{\gamma_f} - \eta\right) = 1. \end{aligned}$$

The last two expressions, together, imply

$$-\eta \le \sigma_{\gamma_f} \le \eta$$

Since $-\eta \le \sigma_{\gamma_f} \le \eta$ define $\sigma_{\gamma_f} \equiv \chi\eta$ for $\chi \in [-1, 1]$. Substitution of the Defender controls into the terminal Hamiltonian gives

$$\mathscr{H}_d(t_f) = \nu \cos\psi_f + \chi\eta \frac{\nu}{R_f} \sin\psi_f - \eta = 0.$$

The Attacker must maximize the Hamiltonian, and thus

$$\cos\psi_f^* = \frac{1}{\sqrt{1 + \frac{\chi^2\eta^2}{R_f^2}}}, \qquad \sin\psi_f^* = \frac{\chi\eta}{R_f\sqrt{1 + \frac{\chi^2\eta^2}{R_f^2}}}.$$

The terminal Hamiltonian becomes

$$\mathscr{H}_d(t_f)^* = \nu\sqrt{1 + \frac{\chi^2\eta^2}{R_f^2}} - \eta = 0.$$

Solving for η:

$$\begin{aligned} \eta &= \nu\sqrt{1 + \frac{\chi^2\eta^2}{R_f^2}} \\ \eta^2 &= \nu^2\left(1 + \frac{\chi^2\eta^2}{R_f^2}\right) \\ \implies \eta &= \pm\frac{\nu R_f}{\sqrt{R_f^2 - \chi^2}} \end{aligned}$$

Recall $\eta \equiv \sigma_\alpha \equiv \frac{\partial V}{\partial \alpha}$; thus an increase in α should give advantage to the Attacker which implies $\eta > 0$. At general time, the Hamiltonian is

$$\begin{aligned} &\mathscr{H}_d(t) = 0 \\ &\quad = -\sigma_R \nu \cos\psi + \frac{\nu^2\chi R_f}{R\sqrt{R_f^2 - \chi^2}} \sin\psi - \frac{\nu R_f}{\sqrt{R_f^2 - \chi^2}} \end{aligned}$$

Again, the Attacker maximizes the Hamiltonian,

$$\begin{aligned} \cos\psi^* &= \frac{-\sigma_R}{\sqrt{\sigma_R^2 + \frac{\chi^2\nu^2R_f^2}{R^2\left(R_f^2 - \chi^2\right)}}}, \\ \sin\psi^* &= \frac{\nu\chi R_f}{R\sqrt{R_f^2 - \chi^2}\sqrt{\sigma_R^2 + \frac{\nu^2\chi^2R_f^2}{R^2\left(R_f^2 - \chi^2\right)}}}. \end{aligned}$$

Substituting back into the Hamiltonian and solving for σ_R:

$$\sigma_R = \pm\sqrt{\frac{R^2R_f^2 - \nu^2\chi^2R_f^2}{R^2\left(R_f^2 - \chi^2\right)}}.$$

Finally, substitution into the equilibrium Attacker control gives

$$\sin\psi^* = \chi\frac{\nu}{R}.$$

□

Note that the form of the Attacker equilibrium control for this scenario is identical to that of the *Game of Angle* scenario.

Lemma 13. The trajectories corresponding to $\chi = \pm 1$ separate the state space into regions of solo capture ($\gamma_f \neq 0$) and dual capture ($\gamma_f = 0$).

Proof. The result follows from substitution of $\chi = 1$ or $\chi = -1$ into (88). □

Lemma 14. Attacker trajectories resulting in dual capture ($\gamma_f = 0$) are straight lines in the (x, y)-plane terminating at a point I' where

$$I' = \begin{bmatrix} I'_x \\ I'_y \end{bmatrix} = \begin{bmatrix} R_f \cos(\beta_0 - \alpha_0) \\ R_f \sin(\beta_0 - \alpha_0) \end{bmatrix}. \tag{89}$$

Proof. The Attacker equilibrium control, (88), is identical in form to that of the *Game of Angle* solution, (74), and thus the Attacker path is a straight line for the same arguments as presented in Lemma 9. The angle $\beta_0 - \alpha_0$ corresponds to the $\gamma = 0$ axis, which, as in Lemma 9, is invariant under equilibrium play. □

Lemma 15. The surfaces separating solo and dual capture are given by the expression

$$w\left(\hat{R}\right) = \pm\frac{\nu^2\alpha}{R_f}, \tag{90}$$

where $\hat{R} = R\cos(\gamma) = R_f + \nu\alpha\sqrt{1 - \frac{\nu^2}{R_f^2}}$ is the polar distance measured along the $\gamma = 0$ axis and w is measured perpendicular to the $\gamma = 0$ axis, and $\hat{R} \in \left[1 + \nu\alpha_0\sqrt{1 - \nu^2}, \infty\right]$.

Proof. As in Lemma 10, the dual capture trajectory terminates in α time (since $\dot{\alpha}^* = -1$ and $\alpha = 0$ in the dual capture scenario). The dual capture trajectories are thus straight lines (due to Lemma 14) of length $\nu\alpha_0$ which terminate at I', as defined in Lemma 14. Consider the upper limit of ψ_f^*, which is given by (88) with $\chi = 1$ to be $\psi^* = \sin^{-1}\left(\frac{\nu}{R_f}\right)$. The corresponding distance perpendicular to the $\gamma = 0$ axis is $w = \sin\sin^{-1}\left(\frac{\nu}{R_f}\right) \cdot \nu\alpha_0 = \frac{\nu^2\alpha_0}{R_f}$. This w corresponds to a position which is $\nu\alpha_0\sqrt{1 - \frac{\nu^2}{R_f}}$ further than R_f, i.e., $\hat{R} =$

$R_f + \nu\alpha_0\sqrt{1 - \frac{\nu^2}{R_f^2}}$. Taking the lower limit of ψ_f^* gives the corresponding negative width. $\square$

We define the regions $\mathscr{R}_{D_1}$ and $\mathscr{R}_{D_2}$ as the sets of states for which the game terminates with $\gamma_f > 0$ (one-on-one with D_1) and $\gamma_f < 0$ (one-on-one with D_2), respectively (c.f. Lemma 11). Similarly, we define the region $\mathscr{R}_{D_{1,2}}$ as the set of states for which the game terminates with $\gamma_f = 0$ which is completely specified by Lemma 10 and Lemmas 13–13.

Theorem 4. For states in the region $\mathscr{R}_{D_{1,2}}$ the equilibrium Attacker heading angle is

$$\psi^* = \gamma + \sin^{-1}\left(\frac{R\sin\gamma}{\nu\alpha}\right) \tag{91}$$

and the Value function is

$$V(\mathbf{x}) = -R_f = \nu\alpha\frac{\sin\psi^*}{\sin\gamma}. \tag{92}$$

Proof. The result follows from Lemmas 12–14 via a geometric proof process similar to Theorem 3. Consider the triangle $\triangle AI'C$, as shown in Fig. 6, where C is the target circle's center. Since the system begins in $\mathscr{R}_{D_{1,2}}$, the scenario terminates with $\alpha_f = \gamma_f = 0$. The time for each Defender to traverse an angle α around the perimeter of the target circle is α, since $\dot{\alpha} = -1$. Therefore, $\overline{AI'} = \nu\alpha$. Using the Law of Sines, the quantities are related as follows

$$\frac{R_f}{\sin(2\pi - \psi)} = \frac{\nu\alpha}{\sin\gamma} = \frac{R}{\sin(\psi - \gamma)}$$
$$\implies \frac{R_f}{-\sin\psi} = \frac{\nu\alpha}{\sin\gamma} = \frac{R}{\sin(\psi - \gamma)}.$$

The second equality may be rearranged to obtain (91). Likewise, the first equality may be rearranged to obtain (92). $\square$

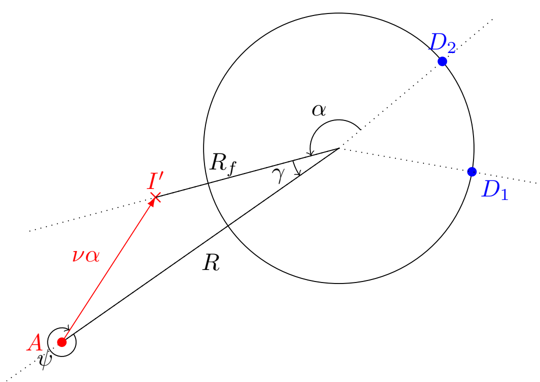

Fig. 6. Illustration of the derivation of the equilibrium Attacker heading and Value function for the two-Defender *Game of Distance* with symmetric termination.

C. Full Solution

The two Defender game is truly three dimensional (in the reduced state space, i.e., R, γ, α). Although one may obtain the equilibrium flowfield over the whole state space by substituting the equilibrium strategies into the kinematics, it is more illustrative to visualize the solution in the (x, y)-plane for a particular α. Figure 7 shows the full solution of the two-Defender one-Attacker game, including all of the separating surfaces, regions, and salient features along with several representative Attacker trajectories. Note that the solution may be generalized to any number of Defenders simply by considering the two Defenders nearest to the Attacker's initial position.

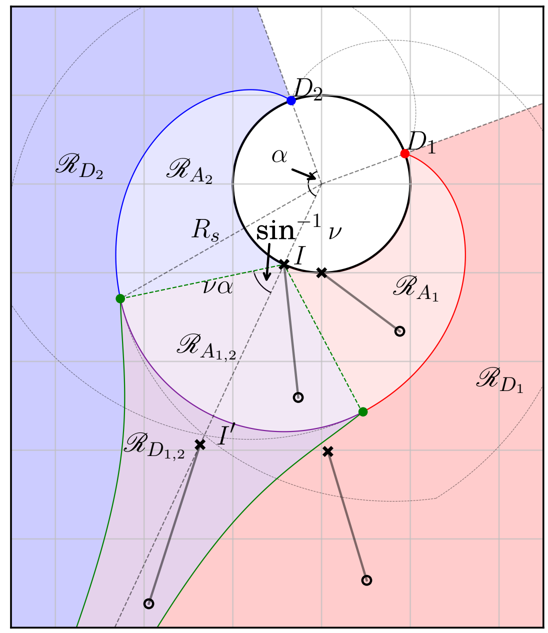

Fig. 7. Separating surfaces for the two Defender game in the realistic plane for $\alpha_0 = \frac{3\pi}{4}$ and $\nu = 0.8$. Representative Attacker trajectories are shown in the symmetric termination regions and Defender 1 regions. Open black circles denote different Attacker initial positions, black $\times$'s denote the corresponding terminal Attacker positions.

D. Alternative Attacker Win Scenario

As in the one-Defender case, we present here the solution of the *Game of Min Time* wherein the cost functional is $\min\max -t_f$, i.e., (34). The analysis follows quite closely with those in the preceding sections. Figure 8 shows the solution with several representative trajectories.

"Cooperation" among the Defenders, i.e., where neither Defender is redundant, only occurs when the Attacker begins on the purple semi-circular section of $\partial\mathscr{R}_{A_1}$ and $\partial\mathscr{R}_{A_2}$. Otherwise, the Attacker plays the single-Defender version of the game with the nearest Defender. The main advantage for having two Defenders is that the state space $\mathscr{R}_{A_1} \cup \mathscr{R}_{A_2}$ is strictly smaller than the single-Defender state space, $\mathscr{R}_A$ (which is true for the *Game of Angle* as well).

IV. Conclusion

The problem of guarding a circular target by patrolling its perimeter was considered. We formulated the one-Defender one-Attacker and two-Defender one-Attacker scenarios as zero-sum differential games with different cost/payoff functionals depending on whether the Attacker could reach the target's perimeter before the Defender(s) could 'lock on'. The analysis formally verifies that the Attacker heading strategy given in the literature for the one-Defender scenario is indeed

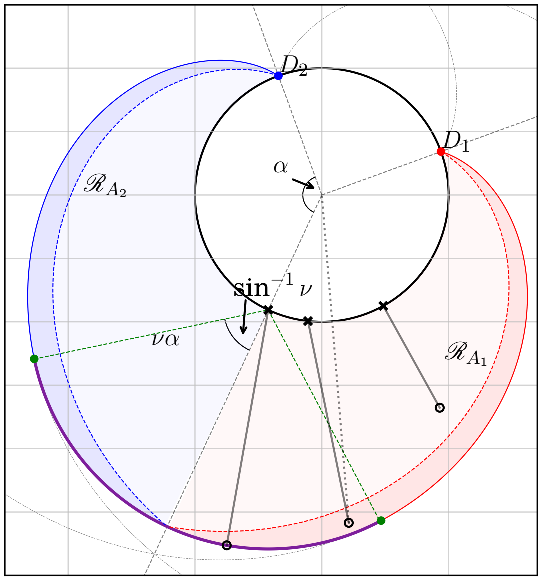

Fig. 8. Two Defender *Game of Min Time* state space for a particular α with $\nu = 0.8$. The 3 Attacker trajectories, left-to-right, are 1) limiting, symmetric termination, 2) evasive (A cannot aim directly at the target circle center), and 3) direct (A aims at the circle center). Initial conditions in the light shaded regions result in direct trajectories, whereas the dark shaded regions represent initial conditions resulting in evasive trajectories.

the saddle-point equilibrium strategy for the games posed here [3]. For the two-Defender scenario, the state space was partitioned into regions based on the equilibrium termination condition. Analytic expressions for the separating surfaces between these regions and Value functions for each case were derived. The Attacker strategy in the Defenders-win, symmetric termination region differs from that of [3], partly due to differences in the termination condition and cost/payoff functional. An alternative scenario in which the Attacker seeks to reach the target in minimum time was also solved for both the one- and two-Defender cases.

Future work on this problem will focus on understanding the impact of these termination conditions on multi-Attacker multi-Defender scenarios.

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