Guarding a Circular Target By Patrolling its Perimeter

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***Abstract*— In this paper, the problem of guarding a circular target wherein the Defender(s) is constrained to move along its perimeter is posed and solved using a differential game theoretic approach. Both the one-Defender and two-Defender scenarios are analyzed and solved. The mobile Attacker seeks to reach the perimeter of the circular target, whereas the Defender(s) seeks to align itself with the Attacker, thereby ending the game. In the former case, the Attacker wins, and the Attacker and Defender play a zero sum differential game where the payoff/cost is the terminal angular separation. In the latter case, the Defender(s) wins, and the Attacker and Defender play a zero sum differential game where the cost/payoff is the Attacker's terminal distance to the target. This formulation is representative of a scenario in which the Attacker inflicts damage on the target as a function of its terminal distance. The state-feedback equilibrium strategies and Value functions for the Attacker-win and Defender(s)-win scenarios are derived for both the one- and two-Defender cases, thus providing a solution to the Game of Degree. Analytic expressions for the separating surfaces between the various terminal scenarios are derived, thus providing a solution to the Game of Kind.**

I. Introduction

The problem of guarding a target has many important applications in real-world defense scenarios. One example is protection of a building's perimeter from mobile infiltrators, which may be considered to be people, ground vehicles, air vehicles, or even certain types of munitions. Isaacs considered such a target guarding problem in his seminal work on differential game theory [1, see Example 1.9.2]. There, the static target was a convex area. Recently, there has been an interest in the defense of a mobile target, usually represented by a point or disk (e.g. [2]).

We pose and analyze the target guarding problem wherein the Defender (D) is constrained to move along the (static) circular target perimeter and the mobile Attacker (A) moves with simple motion. This problem is an instance of the perimeter defense problem presented in [3], [4], [5]; these works establish strategies for individual agents as well as teams of Attackers and Defenders for targets of arbitrary convex shape. One significant difference in the formulation presented here is that we consider the game to terminate when either A reaches the target (A wins), or D becomes aligned with A (D wins). The latter scenario may be thought of as the Defender being able to neutralize the Attacker (at a distance) with a highly directional weapon.

Because the perimeter is a circle and the Defender is constrained to move along the circle, this problem has a strong connection to the *Lady in the Lake* differential game wherein the pursuer runs along the shoreline of a lake (the circle) to try and catch the evader who must swim to the shore from inside the lake to escape [6], [7], [8]. Here, however, we are essentially analyzing the *Lady Outside the Lake* game with the agents' roles reversed. The cost functional, in the case that the Attacker can reach the perimeter, is identical to the *Lady in the Lake* game.

This paper contains the following contributions: (i) the one-on-one Attacker-win and Defender-win scenarios are formulated and solved rigorously using a differential game theoretic approach, verifying the saddle-point equilibrium status of strategies existing in the literature [3]; (ii) analytic expressions for the Value functions are derived for both one-on-one scenarios; (iii) the two-Defender, one-Attacker scenarios are formulated and the equilibrium strategies and Value functions are derived; (iv) the entire state space is partitioned based on all of the different terminal scenarios, and analytic expressions for the separating surfaces are derived. The emphasis is on the analysis and proof methods, which are based on differential game theory, in comparison to the geometric methods used previously [3]. Sections II and III cover the one- and two-Defender cases, respectively. In each of those sections, both the Attacker-win and Defender(s)-win scenarios are formulated and solved. Section IV concludes the work.

II. One Defender

This section formulates the target guarding problem wherein the Defender (D) is constrained to move along the circular target perimeter and the Attacker (A) moves in the plane with simple motion. Figure 1 shows the local coordinate system (black) used in much of the analysis to appear, as well as the global (inertial) (x, y)-coordinate system (green). The following assumptions are made on the problem setup:

Assumption 1. The players' speeds are such that $v_A \leq v_D$.

Assumption 2. The initial separation angle is such that $\theta(t_0) = \theta_0 \in [0, \pi)$.

The (dimensional) kinematics, based on Fig. 1 are

$$\bar{f}\left(\bar{\mathbf{x}}, \bar{u}, \bar{t}\right) = \dot{\bar{\mathbf{x}}} = \begin{bmatrix} \dot{\bar{R}} \\ \dot{\bar{\theta}} \\ \dot{\bar{\beta}} \end{bmatrix} = \begin{bmatrix} -v_A \cos\psi \\ \frac{v_A}{R}\sin\psi - \frac{v_D}{l} \\ \frac{v_D}{l} \end{bmatrix}, \qquad (1)$$

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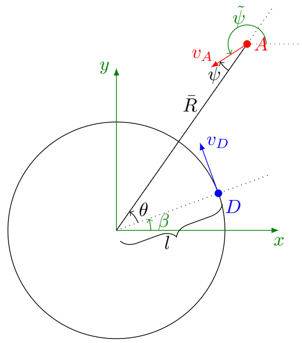

Fig. 1. Circular perimeter patrol with one Defender and one Attacker.

where $\theta \in [-\pi, \pi]$ is the angle of A's position w.r.t. D and $\beta \in [0, 2\pi]$ represents the rotation of D about the circle's center w.r.t. a global (x, y)-plane. With the following definitions,

$$R \equiv \frac{\bar{R}}{l}, \quad t \equiv \frac{v_{D_{\max}}}{l}\bar{t}, \quad u_D \equiv \frac{v_D}{v_{D_{\max}}}, \quad \nu = \frac{v_A}{v_{D_{\max}}},$$

where $v_{D_{\max}}$ is the maximum Defender velocity and the speed ratio $0 < \nu \leq 1$, the kinematics in (1) are non-dimensionalized:

$$f(\mathbf{x}, u, t) = \dot{\mathbf{x}} = \begin{bmatrix} \dot{R} \\ \dot{\theta} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -\nu \cos\psi \\ \nu \frac{1}{R} \sin\psi - u_D \\ u_D \end{bmatrix}. \tag{2}$$

The Defender control lies in the range $u_D \in [-1, 1]$, and the Attacker control lies in the range $\psi \in [-\pi, \pi]$.

We define the *Game of Kind* as the question of whether Attacker can reach the perimeter ($R \to 1$) with non-zero terminal separation angle (Attacker 'wins') or the Defender can drive $\theta \to 0$ before the Attacker reaches the perimeter (Defender 'wins'). In the following sections, the surface separating these two cases is derived and a *Game of Degree* is specified and solved for each case.

Note that if $v_A > v_D$, the Attacker need only come within some distance $l < \hat{R} < \frac{v_A}{v_D}$ wherein the Attacker has the control authority to force $\theta \to \pi$. Similarly, when $v_A \leq v_D$, if at some point $\theta = 0$ the game is over because the Defender has sufficient control authority to keep $\theta = 0$ regardless of the Attacker's control. We assume that if $\theta_f = 0$ the Defender has successfully intercepted the Attacker and thwarted its attack. We refer to the question of whether the Attacker wins (i.e. $\theta_f > 0$) or the Defender wins ($\theta_f = 0$) as the *Game of Kind*.

A. Defender Win Scenario

In this section we are concerned with the *Game of Degree* which takes place when D is able to drive $\theta \to 0$ before A can reach the target. Here, the initial condition of the system lies in the region $\mathscr{R}_D$, which is the region of win for the Defender (see (28)). In this case, it is sensible for the agents to play a zero-sum game over the cost functional

$$J_d = \Phi_d(\mathbf{x}_f, t_f) = -R_f. \tag{3}$$

The negative sign in (3) is present so that the Defender is the minimizing player and the Attacker is the maximizing player. That is, the Attacker seeks to get as close as possible to $R_f = 1$ and the Defender seeks to maximize the terminal distance. We refer to this game as the *Game of Distance*, and denote it with subscript d, in general. If an equilibrium exists, it's Value function is defined as

$$V_d = \min_{u_D(t)} \max_{\psi(t)} J_d = \max_{\psi(t)} \min_{u_D(t)} J_d. \tag{4}$$

The first order necessary conditions for equilibrium will be developed in the subsequent analysis. The terminal constraint for the *Game of Distance* is

$$\phi_d(\mathbf{x}_f, t_f) = \theta_f = 0. \tag{5}$$

The final time, t_f, is the first time for which $\theta(t) = 0$. Thus, the Terminal Surface is defined as the set of states satisfying (5)

$$\mathscr{T}_d = \{\mathbf{x} \mid R \geq 1 \text{ and } \theta = 0\}. \tag{6}$$

Assumptions 1 and 2 are retained for this analysis.

1) First Order Necessary Conditions for Optimality: The kinematics remain unchanged from the previous analysis; the Hamiltonian for the *Game of Distance* is

$$\mathscr{H}_d = -\sigma_R \nu \cos\psi + \sigma_\theta \left(\nu \frac{1}{R} \sin\psi - u_D\right) + \sigma_\beta u_D, \tag{7}$$

where $\sigma \equiv \begin{bmatrix} \sigma_R & \sigma_\theta & \sigma_\beta \end{bmatrix}^\top$ is the adjoint vector for the *Game of Distance*. The Hamiltonian is a separable function of the controls u_D and ψ, and thus *Isaacs' condition* [8], [1] holds:

$$\min_{u_D(t)} \max_{\psi(t)} \mathscr{H} = \max_{\psi(t)} \min_{u_D(t)} \mathscr{H}.$$

The equilibrium adjoint dynamics are given by

$$\dot{\sigma}_R = -\frac{\partial \mathscr{H}_d}{\partial R} = \nu \sigma_\theta \frac{1}{R^2} \sin\psi, \tag{8}$$

$$\dot{\sigma}_\theta = -\frac{\partial \mathscr{H}_d}{\partial \theta} = 0, \tag{9}$$

$$\dot{\sigma}_\beta = -\frac{\partial \mathscr{H}_d}{\partial \beta} = 0. \tag{10}$$

The terminal adjoint values are obtained from the transversality condition [9]

$$\begin{aligned} \sigma^\top(t_f) &= \frac{\partial \Phi_d}{\partial \mathbf{x}_f} + \eta \frac{\partial \phi_d}{\partial \mathbf{x}_f} \\ &= \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} + \eta \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \\ \sigma_{R_f} &= -1 \\ \implies \quad \sigma_{\theta_f} &= \eta \\ \sigma_{\beta_f} &= 0. \end{aligned} \tag{11}$$

Therefore, with (9)–(11), the following hold

$$\sigma_\theta(t) = \eta, \qquad \forall t \in [t_0, t_f] \tag{12}$$

$$\sigma_\beta(t) = 0, \qquad \forall t \in [t_0, t_f]. \tag{13}$$

Once again, since $\sigma_\beta(t) = 0$ for all $t \in [t_0, t_f]$, the state component β has no effect on the equilibrium trajectory or the equilibrium control strategies. The terminal Hamiltonian satisfies [9]

$$\mathscr{H}_d(t_f) = -\frac{\partial \Phi_d}{\partial t_f} - \eta \frac{\partial \phi_d}{\partial t_f} = 0, \tag{14}$$

and $\frac{\mathrm{d}\mathscr{H}_d}{\mathrm{d}t} = 0$, so $\mathscr{H}_d(t) = 0$ for all $t \in [t_0, t_f]$.

The equilibrium control actions of the Attacker and Defender maximize and minimize (7), respectively: $\mathscr{H}_d^* = \max_\psi \min_{u_D} \mathscr{H}_d$. In order to maximize (7) (with (12)), the vector $\begin{bmatrix}\cos\psi & \sin\psi\end{bmatrix}$ must be parallel to the vector $\begin{bmatrix}\sigma_R & \frac{\eta}{r}\end{bmatrix}$, giving

$$\cos\psi^* = \frac{-\sigma_R}{\sqrt{\sigma_R^2 + \frac{\eta^2}{R^2}}}, \qquad \sin\psi^* = \frac{\eta}{R\sqrt{\sigma_R^2 + \frac{\eta^2}{R^2}}}. \tag{15}$$

If $\eta < 0$, this implies $\sin\psi^* < 0$ due to (15). However, this would mean the Attacker has a component of its motion that points *towards* the Defender (see, e.g., Fig. 1). Thus, it must be the case that $\eta > 0$. In order to minimize (7) (with (12)), the Defender's control must satisfy

$$u_D^* = \operatorname{sign}\eta = 1, \tag{16}$$

since $\eta > 0$.

Substituting the equilibrium controls, (15) and (16), into the Hamiltonian, (7), and evaluating at final time with (11) and (14) gives

$$\begin{aligned}\mathscr{H}_d^*(t_f) = 0 &= \frac{\nu\sigma_{R_f}^2}{\sqrt{\sigma_{R_f}^2 + \frac{\eta^2}{R_f^2}}} + \frac{\nu\eta^2}{R_f^2\sqrt{\sigma_{R_f}^2 + \frac{\eta^2}{R_f^2}}} - \eta \\ \implies \eta &= \pm\nu R_f\sqrt{\frac{1}{R_f^2 - \nu^2}}.\end{aligned}$$

Since $\eta > 0$, we have

$$\eta = \nu R_f\sqrt{\frac{1}{R_f^2 - \nu^2}}. \tag{17}$$

2) Solution Characteristics: An expression for σ_R is obtained by considering the Hamiltonian at a general time, making the same substitutions as before, with the additional substitution of (17):

$$\begin{aligned}\mathscr{H}_d^*(t) = 0 &= \nu\sqrt{\sigma_R^2 + \frac{\eta^2}{R^2}} - \eta \\ \implies \sigma_R &= \pm\sqrt{\frac{\eta^2}{\nu^2} - \frac{\eta^2}{R^2}} \\ &= \pm\frac{R_f}{R}\sqrt{\frac{R^2 - \nu^2}{R_f^2 - \nu^2}}.\end{aligned}$$

Since $\sigma_{R_f} < 0$ (due to (11)) and $\dot{\sigma}_R > 0$ (due to (8) with (15) and $\eta > 0$) it must be that $\sigma_R(t) < 0$ for all $t \in [t_0, t_f]$, thus

$$\sigma_R = -\frac{R_f}{R}\sqrt{\frac{R^2 - \nu^2}{R_f^2 - \nu^2}}. \tag{18}$$

The retrograde equilibrium kinematics can be obtained by substituting the equilibrium controls, (15) and (16), along with the adjoints, (12), (13), and (18), into (2) which yields

$$\mathring{R}^* = \nu\sqrt{1 - \frac{\nu^2}{R^2}}, \qquad \mathring{\theta}^* = 1 - \frac{\nu^2}{R^2}, \tag{19}$$

with the following boundary conditions

$$R(t_f) > 1, \qquad \theta(t_f) = 0. \tag{20}$$

Note that both $\mathring{R}$ and $\mathring{\theta}$ are monotonically increasing according to (19). Consider the differential equation obtained by dividing the equations in (19)

$$\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}\theta} &= \frac{\nu}{\sqrt{1 - \frac{\nu^2}{R^2}}} \\ \implies \nu&\left[\sqrt{\frac{R^2}{\nu^2} - 1} + \sin^{-1}\left(\frac{\nu}{R}\right)\right]_{R_f}^{R} = \nu\left(\theta - \theta_f\right).\end{aligned}$$

Define

$$g(R) = \sqrt{\frac{R^2}{\nu^2} - 1} + \sin^{-1}\left(\frac{\nu}{R}\right), \tag{21}$$

$$\begin{aligned}&\implies \nu\left(g(R) - g(R_f)\right) = \nu\left(\theta - \theta_f\right) \\ &\implies \theta\left(R; R_f, \theta_f\right) = g(R) - g(R_f) + \theta_f, \quad \theta_f \le \theta < \pi.\end{aligned} \tag{22}$$

Setting $\theta_f = 0$ in (22) (i.e., $\theta(R; R_f, 0)$) describes the equilibrium flow field for the *Game of Distance* (i.e., assuming the Defender can drive $\theta \to 0$ before the Attacker can reach the target). The curve in (22) is the involute of a circle of radius ν.

Up until now, we have considered θ to be in the range $[0, \pi)$, however, the results apply to the range $(-\pi, 0]$ with some slight modification.

Lemma 1. The surface

$$\mathscr{D} \equiv \{\mathbf{x} \mid \theta = \pi\}, \tag{23}$$

is a Dispersal Surface (c.f. [1]) wherein the Defender can choose either $u_D = 1$ or $u_D = -1$ and both choices are optimal. Furthermore, when $\theta < 0$, the equilibrium control are given by $u_D^* = -1$ and $\sin\psi^* < 0$.

Proof. Consider a state $\mathbf{x}_{\mathscr{D}} = (R_0, \theta_0, \beta_0) \in \mathscr{D}$. The system (19) describes the evolution of the R and θ in backwards time from $R_f > 1$ and $\theta_f = 0$, assuming $\eta > 0$. For a particular $\mathbf{x}_f$, where $\theta_f = 0$, (19) may be integrated back to $\mathbf{x}_{\mathscr{D}}$. A symmetric solution can be constructed by switching the sign of η and integrating the retrograde kinematics back to $\mathbf{x}_{\mathscr{D}}$. Now, let $\eta < 0$; then $\sin\psi^* < 0$ from (15), and $u_D^* = -1$ from (16). Substitution into the Hamiltonian at final time yields $\eta = -\nu R_f\sqrt{\frac{1}{R_f^2 - \nu^2}}$. Substituting all of these into the Hamiltonian at general time yields the same expression for σ_R as in (18). Then, from (2), the retrograde kinematics are

$$\mathring{R} = \nu\sqrt{1 - \frac{\nu^2}{R^2}}, \qquad \mathring{\theta} = \frac{\nu^2}{R^2} - 1.$$

Clearly, these are the same kinematics as in (19) except the sign of $\mathring{\theta}$ is reversed. Thus, both sets of equilibrium kinematics can be integrated back from $\mathbf{x}_f$ to reach $\mathbf{x}_{\mathscr{D}}$, noting that $-\pi$ and π are equivalent. Note this method for proving the presence of a Dispersal Surface is similar to the one used for a problem with similar dynamics in [10]. □

As a consequence, Assumption 2 may be relaxed, and the state space may be expanded to $\theta \in [-\pi, \pi]$.

Theorem 1 (*Game of Distance* Solution). The equilibrium state feedback control strategies for the *Game of Distance* are given by

$$\psi^* = \operatorname{sign}(\theta) \sin^{-1}\left(\frac{\nu}{R}\right), \qquad u_D^* = \operatorname{sign}(\theta). \tag{24}$$

The Value of the game is

$$V_d(R, \theta) = -R_f = -g^{-1}\left(g(R) - \theta\right). \tag{25}$$

Proof. The expression for ψ^* is obtained by substituting (17) and (18) into (15), taking into account the sign of θ (due to Lemma 1). Similarly, the Defender strategy is given by (16), accounting for Lemma 1. The corresponding form of (22) for the *Game of Distance* is

$$\theta\left(R; R_f\right) = g(R) - g(R_f). \tag{26}$$

Thus, (25) is obtained by rearranging this expression and solving for R_f, with $g(\cdot)$ defined as in (21). □

The Value function does not have a closed form analytic expression since g^{-1} cannot be expressed in closed form.

The limiting case for the *Game of Distance* is one in which $R_f \to 1$; thus the surface

$$\theta_{GoK}(R) = g(R) - g(1) \tag{27}$$

partitions the state space into regions of win for the Defender and Attacker, respectively,

$$\mathscr{R}_D = \{\mathbf{x} \mid |\theta| \leq \theta_{GoK}(R)\} \tag{28}$$
$$\mathscr{R}_A = \{\mathbf{x} \mid |\theta| > \theta_{GoK}(R)\}. \tag{29}$$

B. Attacker Win Scenario

In the region of the state space in which the Attacker 'wins', we consider a *Game of Degree* wherein the players max/min the terminal separation angle; we refer to this as the *Game of Angle*. The cost/payoff functional is given as

$$J = \Phi\left(\mathbf{x}_f, t_f\right) = \theta_f. \tag{30}$$

The Attacker seeks to maximize the terminal separation angle whereas the Defender seeks to minimize.

Theorem 2 (*Game of Angle* Solution). The equilibrium state feedback strategies for the *Game of Angle* match those of the *Game of Distance*, i.e., are given by (24). The Value function is given by

$$V(R, \theta) = \theta_f = \theta - g(R) + g(1). \tag{31}$$

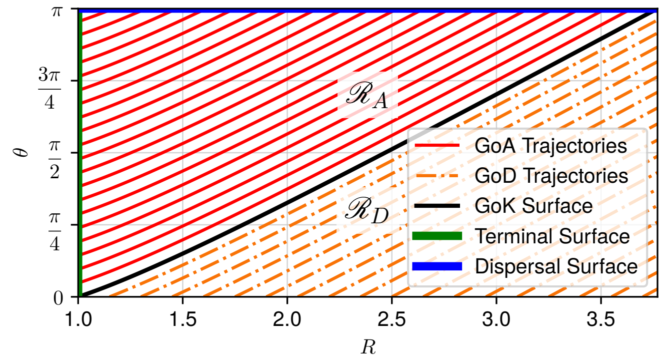

Fig. 2. Full equilibrium flow field with $\nu = 0.8$

Proof. This proof is based upon substitution of the proposed equilibrium strategies and Value function into the Hamilton-Jacobi-Isaacs (HJI) equation [1],

$$\min_{u_D} \max_{\psi} \left\{ l\left(\mathbf{x}, u_D, \psi, t\right) + \frac{\partial V}{\partial t} + V_{\mathbf{x}} \cdot f\left(\mathbf{x}, u_D, \psi, t\right) \right\} = 0, \tag{32}$$

where $V_{\mathbf{x}}$ is the vector $\begin{bmatrix} \frac{\partial V}{\partial R} & \frac{\partial V}{\partial \theta} & \frac{\partial V}{\partial \beta} \end{bmatrix}^\top$, and l represents an integral cost component. First, note that the cost, (30), has no integral component, and thus $l = 0$. Also, the proposed Value function, (31) is not an explicit function of time and thus $\frac{\partial V}{\partial t} = 0$. The vector $V_{\mathbf{x}}$ is obtained by differentiating (31) w.r.t. each state,

$$V_{\mathbf{x}} = \begin{bmatrix} \frac{-\sqrt{R^2 - \nu^2}}{R\nu} & 1 & 0 \end{bmatrix}.$$

The (forward) equilibrium dynamics, f, are given by the negative of (19). Substituting all of these expressions into (32) gives

$$\frac{\partial}{\partial R}\dot{R} + \frac{\partial V}{\partial \theta}\dot{\theta} = \left(\frac{-\sqrt{R^2 - \nu^2}}{R\nu}\right)\left(-\nu\sqrt{1 - \frac{\nu^2}{R^2}}\right) + \left(\frac{\nu^2}{R^2 - 1}\right) = 0.$$

The proposed Value function is continuous and continuously differentiable (except on the Dispersal Surface, $\mathscr{D}$), and it satisfies the HJI hyperbolic PDE. □

C. Full Equilibrium Flow Field

With the analysis in Sections II-A and II-B, the entire (usable) state space can be filled with equilibrium trajectories. Figure 2 shows (22) and (26) in the Attacker win and lose regions, respectively.

Lemma 2. The Attacker's equilibrium trajectory is a straight line in the inertial (non-rotating) (x, y)-plane.

Proof. Consider Fig. 1 which shows the Attacker's heading angle, $\tilde{\psi}$, w.r.t. the inertial (x, y)-plane. The following relation holds

$$\tilde{\psi} = \beta + \theta + \pi - \psi$$

Thus, the time derivative of the global Attacker heading angle is given as

$$\dot{\tilde{\psi}} = \dot{\beta} + \dot{\theta} - \dot{\psi}$$

Substituting (24) and (19) into the above gives

$$\begin{aligned}\dot{\tilde{\psi}} &= 1 + \frac{\nu^2}{R^2} - 1 - \frac{\partial}{\partial t}\sin^{-1}\left(\frac{\nu}{R}\right) \\ &= \frac{\nu^2}{R^2} - \left(\frac{-1}{\sqrt{1-\frac{\nu^2}{R^2}}}\right)\left(\frac{\nu}{R^2}\right)\dot{R} \\ &= \frac{\nu^2}{R^2} + \left(\frac{1}{\sqrt{1-\frac{\nu^2}{R^2}}}\right)\left(\frac{\nu}{R^2}\right)\left(-\nu\sqrt{1-\frac{\nu^2}{R^2}}\right) \\ &= 0.\end{aligned}$$

Because $\dot{\tilde{\psi}} = 0$, the global Attacker heading angle is constant, and thus the Attacker's path is a straight-line in the inertial (x, y)-plane. $\square$

III. Two Defenders

In this section, we consider the circular target guarding game with two Defenders, D_1 and D_2, with the following assumption:

Assumption 3. The two Defenders share the same maximum speed: $v_{D_{1_{\max}}} = v_{D_{2_{\max}}} = v_{D_{\max}}$.

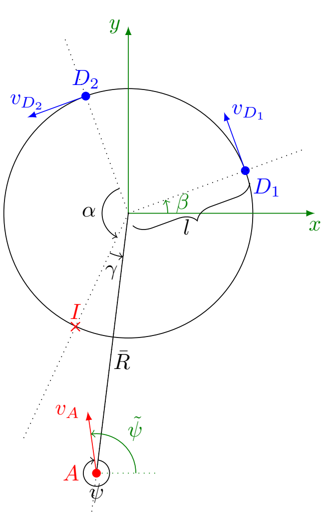

Fig. 3. Circular perimeter patrol with two Defenders and one Attacker.

The scenario is depicted in Fig. 3, and the (nondimensional) kinematics of the system are given as

$$f(\mathbf{x}, \mathbf{u}, t) = \dot{\mathbf{x}} = \begin{bmatrix}\dot{R} \\ \dot{\gamma} \\ \dot{\alpha} \\ \dot{\beta}\end{bmatrix} = \begin{bmatrix}-\nu\cos\psi \\ \frac{\nu}{R}\sin\psi - \frac{1}{2}\left(u_{D_1} + u_{D_2}\right) \\ \frac{1}{2}\left(u_{D_1} - u_{D_2}\right) \\ u_{D_1}\end{bmatrix}. \tag{33}$$

The angle α is measured from D_2 to the angular bisector (on the side of A) of the positions of D_1 and D_2. Similarly, the angle γ is measured as A's angular offset w.r.t. this bisector.

Assumption 4. The relative angular position of the Attacker is bounded such that $-\alpha \leq \gamma \leq \alpha$.

Although we impose Assumption 4, it is of little consequence since the forthcoming solution would still apply for γ outside this range by, for example, switching the designation of D_1 and D_2. Just as in the analysis of the one-on-one game, there are three "games" or questions of interest: 1) can the Attacker reach the target (the *Game of Kind*), 2) what is the equilibrium terminal angular separation between the Attacker and the closest Defender (the *Game of Angle*), and 3) what is the equilibrium terminal distance from the target center (the *Game of Distance*).

Note that the rotation of the system w.r.t. the global x-axis, β, has no effect on the optimality of the trajectories as in the one-on-one analysis and is therefore omitted in the following.

A. Game of Degree When Attacker Wins

Here, we consider the *Game of Angle* which applies to the scenario when the Attacker is able to reach the target ($R_f = 1$). The cost functional is given as

$$J = \Phi\left(\mathbf{x}_f, t_f\right) = \alpha_f - |\gamma_f|, \tag{34}$$

and we seek the Value of the game:

$$V(\mathbf{x}) = \min_{\mathbf{u}(t)}\max_{\psi(t)} J = \max_{\psi(t)}\min_{\mathbf{u}(t)} J. \tag{35}$$

This game terminates when the following condition is satisfied

$$\phi\left(\mathbf{x}_f, t_f\right) = R_f - 1 = 0. \tag{36}$$

1) First Order Necessary Conditions for Optimality: First, form the Hamiltonian as

$$\begin{aligned}\mathscr{H} = -\lambda_R\nu\cos\psi + \lambda_\gamma\left(\frac{\nu}{R}\sin\psi - \frac{1}{2}\left(u_{D_1} + u_{D_2}\right)\right) \\ +\lambda_\alpha\frac{1}{2}\left(u_{D_1} - u_{D_2}\right).\end{aligned} \tag{37}$$

The equilibrium adjoint dynamics obey [9]

$$\dot{\lambda}_R = -\frac{\partial\mathscr{H}}{\partial R} = \lambda_\gamma\frac{\nu}{R^2}\sin\psi, \tag{38}$$

$$\dot{\lambda}_\gamma = -\frac{\partial\mathscr{H}}{\partial\gamma} = 0, \tag{39}$$

$$\dot{\lambda}_\alpha = -\frac{\partial\mathscr{H}}{\partial\alpha} = 0. \tag{40}$$

From the transversality condition [9], the equilibrium terminal adjoint values satisfy

$$\begin{aligned} &\lambda^\top(t_f) = \frac{\partial\Phi}{\partial\mathbf{x}_f} + \mu\frac{\partial\phi}{\partial\mathbf{x}_f} \\ \implies\quad &\lambda_R(t_f) = \mu, \\ &\lambda_\gamma(t_f) = -\operatorname{sign}(\gamma_f), \\ &\lambda_\alpha(t_f) = 1.\end{aligned} \tag{41}$$

Because $\dot{\lambda}_\gamma = \dot{\lambda}_\alpha = 0$ we have $\lambda_\gamma(t) = -\operatorname{sign}(\gamma_f)$ and $\lambda_\alpha(t) = 1$ for all $t \in [t_0, t_f]$.

2) Solution Characteristics:

Lemma 3. For games terminating with $\gamma_f \neq 0$, the game's Value function and optimal strategies are that of the one-on-one game: $V = \alpha - |\gamma| - g(R) + g(1)$, and $\psi^* = \sin^{-1}\left(-\frac{\nu}{R}\operatorname{sign}(\gamma_f)\right)$. The second Defender is redundant.

Proof. Suppose that $\gamma_f < 0$; substituting the corresponding λ_γ and λ_α values into (37) gives

$$\mathscr{H} = -\lambda_R \nu \cos\psi - \frac{\nu}{R}\sin\psi - u_{D_2}. \tag{42}$$

Note that u_{D_1} does not appear in (42) and thus D_1 has no effect on the optimality of the trajectory and is therefore redundant. Again, since the final time is free, the Hamiltonian, at terminal time, is subject to (14) [9]; that is, $\mathscr{H}(t_f) = 0$. Since (33) are autonomous, we have $\mathscr{H}(t) = 0$ for all $t \in [t_0, t_f]$. Therefore, (42) is identical to the Hamiltonian for the one-on-one case between the Attacker and D_2. Furthermore, the terminal condition is the same, and the cost functional is identical since $\theta = \alpha - |\gamma| = J$, in this case. Thus, the Value function for the one-on-one case, (31), and the equilibrium Attacker heading control, (24) are the solution for this game (making the appropriate substitution of $\theta = \alpha - |\gamma|$). The $-\operatorname{sign}$ in the ψ^* expression, in this case, accounts for the case when $\gamma_f > 0$ in which the game plays out between the Attacker and D_1, by symmetry. In that case, the scenario is a mirror image of Fig. 1 and the sign of u_{D_1} is reversed (i.e., D_1 moves clockwise) as is the sign of $\sin\psi^*$. □

Since $\gamma_f \neq 0$ corresponds to either one-on-one game, we focus our attention on the case when $\gamma_f = 0$. When $\gamma_f = 0$, the Attacker terminates at a position which is equidistant from the two defenders. Note that, according to (41), $\lambda_\gamma(t_f) = \lambda_{\gamma_f}$ is undefined when $\gamma_f = 0$. As before, the Defenders seek to minimize the Hamiltonian, (37):

$$\begin{aligned} u_{D_1}^*, u_{D_2}^* &= \underset{u_{D_1}, u_{D_2}}{\arg\min}\, \mathscr{H} \\ &= \underset{u_{D_1}, u_{D_2}}{\arg\min}\, u_{D_1}\left(1 - \lambda_\gamma\right) + u_{D_2}\left(-1 - \lambda_\gamma\right). \end{aligned} \tag{43}$$

Now, according to (43), if $\lambda_\gamma > 1$ or $\lambda_\gamma < -1$ then $u_{D_1}^* = u_{D_2}^*$ which means the Defenders should move in the same direction. However, if this were the case then $\dot{\alpha} = 0$ which is clearly undesirable since α appears in the cost, J. Thus the value of λ_γ is bounded:

$$-1 \leq \lambda_\gamma \leq 1. \tag{44}$$

By inspection, it is clear that the Defenders should seek to minimize $\dot{\alpha}$ which occurs for

$$u_{D_1}^* = -1, \quad u_{D_2}^* = 1. \tag{45}$$

substituting in (41) and (45) into (37) leads to an expression for μ:

$$\begin{aligned} \mathscr{H}(t) = 0 &= \nu\sqrt{\lambda_R^2 + \frac{\lambda_\gamma^2}{R^2}} - 1 \\ \implies \lambda_R &= \pm\sqrt{\frac{1}{\nu^2} - \frac{\lambda_\gamma^2}{R^2}}. \end{aligned}$$

Since $\dot{R}_f \propto \cos\psi_f \propto \nu$ it must be that $\lambda_{R_f}, \nu < 0$ in order for the state of the system to penetrate the boundary. In order to maximize the Hamiltonian, it must be that $\sin\psi^* \propto \lambda_\gamma$; thus, from (38), $\dot{\lambda}_R(t) < 0$ for all $t \in [t_0, t_f]$. Therefore, $\lambda_R(t) < 0$ for all $t \in [t_0, t_f]$, which leads to

$$\lambda_R = -\sqrt{\frac{1}{\nu^2} - \frac{\lambda_\gamma^2}{R^2}}. \tag{46}$$

Lemma 4. For games terminating with $\gamma_f = 0$, the equilibrium heading angle is

$$\psi^* = \sin^{-1}\left(\lambda_\gamma \frac{\nu}{R}\right), \tag{47}$$

and is bounded by $-\sin^{-1}\left(\frac{\nu}{R}\right) \leq \psi^* \leq \sin^{-1}\left(\frac{\nu}{R}\right)$.

Proof. Substituting (46) with (45) into (37) gives

$$\mathscr{H} = 0 = -\nu\sqrt{\frac{1}{\nu^2} - \frac{\lambda_\gamma^2}{R^2}}\cos\psi + \lambda_\gamma \frac{\nu}{R}\sin\psi. \tag{48}$$

The Attacker seeks to maximize the Hamiltonian, and thus

$$\cos\psi^* = -\sqrt{1 - \frac{\nu^2\lambda_\gamma^2}{R^2}}, \quad \sin\psi^* = \lambda_\gamma \frac{\nu}{R}, \tag{49}$$

and $-1 \leq \lambda_\gamma \leq 1$ according to (44), hence $-\frac{\nu}{R} \leq \sin\psi^* \leq \frac{\nu}{R}$. □

Lemma 5. The trajectories corresponding to $\lambda_\gamma = \pm 1$ separate the state space into regions of asymmetric termination ($\gamma_f \neq 0$) and symmetric termination ($\gamma_f = 0$).

Proof. Suppose $\lambda_\gamma = 1$, then the Attacker's equilibrium strategy is identical to the one-on-one game with D_2 (c.f. (24)). The trajectory is a straight line in the global (x, y)-frame since the one-on-one game Attacker trajectories are straight (due to Lemma 2). Trajectories with $\lambda_\gamma < 1$ lie on one side of this surface and one-on-one trajectories (against D_2) lie on the other side. □

Lemma 6. Attacker trajectories resulting in symmetric termination ($\gamma_f = 0$) are straight lines in the (x, y)-plane terminating at a point I, where

$$I = \begin{bmatrix} I_x & I_y \end{bmatrix}^\top = \begin{bmatrix} \cos\left(\beta_0 - \alpha_0\right) & \sin\left(\beta_0 - \alpha_0\right) \end{bmatrix}^\top. \tag{50}$$

Proof. The proof for straight-line Attacker trajectories follows along the same steps as Lemma 2 and is thus omitted. For symmetric termination, the state of the system lies at $R = 1$ and $\gamma = 0$. The $\gamma = 0$ angle corresponds to $\beta - \alpha$. Because $u_{D_1}^* = -1$ and $u_{D_2}^* = 1$ (due to (49)) we have $\dot{\alpha} = -1 = \dot{\beta}$ and thus the angle $\beta - \alpha$ is invariant in the global (x, y)-plane. □

Lemma 7. For symmetric termination ($\gamma_f = 0$), the separating surface of the *Game of Kind* in the global (x, y)-plane is given by a circular arc centered I with radius $\nu\alpha_0$ whose bounds are defined by $\sin^{-1}(-\nu)$ and $\sin^{-1}(\nu)$ relative to the $\gamma = 0$ axis.

Proof. Symmetric termination trajectories terminate at I, defined and according to Lemma 6. The limiting case occurs when the Attacker reaches the target circle at the exact moment in which the Defenders reach I (i.e. $\alpha_f \to 0$). Due to (45), we have $\dot{\alpha} = -1$. Therefore, the Defenders reach $\alpha = 0$ in α_0 time. Symmetric termination trajectories may thus extend from I for a maximum distance of $\nu\alpha_0$; beyond this distance, the Attacker cannot reach the target. The Attacker trajectories are straight, also due to Lemma 6, thus the *Game of Kind* surface is a circular arc. The bounds of the circular arc are given directly by the range of ψ_f^* which is obtained by substituting $R = 1$ into (47) and applying the bounds stated in Lemma 4. □

The regions $\mathscr{R}_{a_1}$ and $\mathscr{R}_{a_2}$ are the sets of states for which the game terminates with $\gamma_f > 0$ (one-on-one with D_1) and $\gamma_f < 0$ (one-on-one with D_2), respectively (c.f. Lemma 3). Similarly, the region $\mathscr{R}_{a_{1,2}}$ is the set of states for which the game terminates with $\gamma_f = 0$ and is completely specified by Lemmas 5–7. The polar distance at which the *Game of Kind* surface switches from the one-on-one surface, governed by (27), and the two-on-one surface, described in Lemma 7, is given by

$$R_s = +\sqrt{\nu^2\alpha^2 + 1 + 2\nu\alpha\sqrt{1-\nu^2}}, \tag{51}$$

which is derived from the Law of Cosines (see Fig. 4).

Theorem 3. In the region $\mathscr{R}_{a_{1,2}}$, the equilibrium Attacker heading angle is given by

$$\psi^* = \sin^{-1}\left(\frac{\sin\gamma}{p}\right), \tag{52}$$

and the associated Value function is

$$V(\mathbf{x}) = \alpha_f = \alpha - \frac{p}{\nu}, \tag{53}$$

where

$$p = +\sqrt{R^2 + 1 - 2R\cos\gamma}.$$

Proof. Consider the triangle formed by the Attacker's position, the target circle center, and the point I as defined in (50). By construction, the Attacker starts in $\mathscr{R}_{a_{1,2}}$ and its equilibrium trajectory must terminate at I due to Lemma 6. Let the distance traveled from A_0 to I be p, which can be obtained from the Law of Cosines (as defined above). Then, (52) can be obtained from the Law of Sines. The time taken to traverse this path is p/ν, and $\dot{\alpha} = -1$ (due to (45)), thus (53) follows. □

B. Game of Degree When Attacker Loses

In this section, we focus on the *Game of Distance* which applies to the scenario when A is not able to reach the target before one or both Defenders can align with A (i.e. $\alpha - |\gamma| = 0$). The cost functional is the same as in the one-on-one case, i.e., (3). This game terminates when the following condition is satisfied

$$\phi(\mathbf{x}_f, t_f) = \alpha_f - |\gamma_f| = 0. \tag{54}$$

The formal analysis of this can be carried out in much the same way as in Section III-A and is omitted for space.

1) Solution Characteristics:

Lemma 8. For games terminating with $\gamma_f \neq 0$, the game's Value function and optimal strategies correspond to the one-on-one game (c.f. Theorem 1). The second Defender is redundant.

Proof. The proof is similar to that of Lemma 3 in that the Hamiltonian is formed and a particular sign of γ_f is assumed, which results in reduction to the one-on-one Hamiltonian with identical cost and terminal boundary condition. These details are omitted for space. □

Lemma 9. For games terminating with $\gamma_f = 0$, the equilibrium Attacker heading angle is

$$\psi^* = \sin^{-1}\left(\chi\frac{\nu}{R}\right), \qquad \chi \in [-1, 1]. \tag{55}$$

Proof. The proof is similar to that of Lemma 4, but with the associated first order necessary conditions for the *Game of Distance*, which are omitted. □

Lemma 10. The trajectories corresponding to $\chi = \pm 1$ separate the state space into regions of solo capture ($\gamma_f \neq 0$) and dual capture ($\gamma_f = 0$).

Proof. The result follows from substitution of $\chi = 1$ or $\chi = -1$ into (55). □

Lemma 11. Attacker trajectories resulting in dual capture ($\gamma_f = 0$) are straight lines in the (x, y)-plane terminating at a point I' where

$$I' = \begin{bmatrix} I'_x \\ I'_y \end{bmatrix} = \begin{bmatrix} R_f \cos(\beta_0 - \alpha_0) \\ R_f \sin(\beta_0 - \alpha_0) \end{bmatrix}. \tag{56}$$

Proof. The angle $\beta_0 - \alpha_0$ corresponds to the $\gamma = 0$ axis, which, as in Lemma 6, is invariant. Proving the straightness of the trajectories follows a similar process as shown in Lemma 2 and is omitted. □

Lemma 12. The surfaces separating solo and dual capture are given by the expression

$$w\left(\hat{R}\right) = \pm\frac{\nu^2\alpha}{R_f}, \tag{57}$$

where $\hat{R} = R\cos(\gamma) = R_f + \nu\alpha\sqrt{1 - \frac{\nu^2}{R_f^2}}$ is the polar distance measured along the $\gamma = 0$ axis and w is measured perpendicular to the $\gamma = 0$ axis, and $\hat{R} \in \left[1 + \nu\alpha_0\sqrt{1-\nu^2}, \infty\right]$.

Proof. As in Lemma 7, the dual capture trajectory terminates in α time (since $\dot{\alpha}^* = -1$ and $\alpha = 0$ in the dual capture scenario). The dual capture trajectories are thus straight lines (due to Lemma 11) of length $\nu\alpha_0$ which terminate at I', as defined in Lemma 11. Consider the upper limit of ψ_f^*, which is given by (55) with $\chi = 1$ to be $\psi^* = \sin^{-1}\left(\frac{\nu}{R_f}\right)$. The corresponding distance perpendicular to the $\gamma = 0$ axis is $w = \sin\sin^{-1}\left(\frac{\nu}{R_f}\right) \cdot \nu\alpha_0 = \frac{\nu^2\alpha_0}{R_f}$. This w corresponds to a position which is $\nu\alpha_0\sqrt{1 - \frac{\nu^2}{R_f}}$ further than R_f, i.e.,

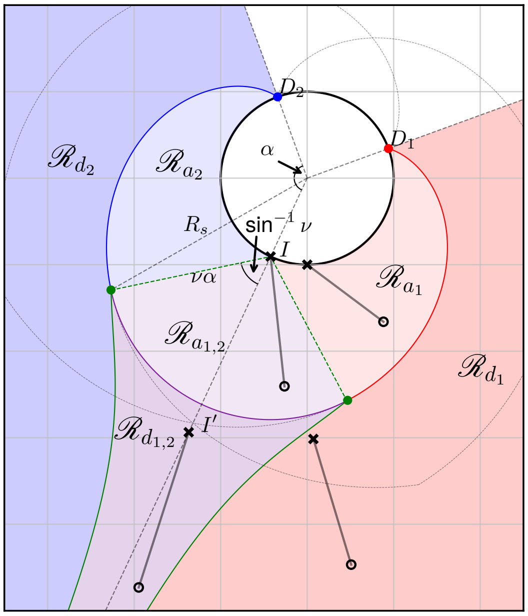

Fig. 4. Separating surfaces for the two Defender game in the realistic plane for $\alpha_0 = \frac{3\pi}{4}$ and $\nu = 0.8$. Representative Attacker trajectories are shown in the symmetric termination regions and Defender 1 regions. Open black circles denote different Attacker initial positions, black $\times$'s denote the corresponding terminal Attacker positions.

$\hat{R} = R_f + \nu\alpha_0\sqrt{1 - \frac{\nu^2}{R_f^2}}$. Taking the lower limit of ψ_f^* gives the corresponding negative width. $\square$

We define the regions $\mathscr{R}_{d_1}$ and $\mathscr{R}_{d_2}$ as the sets of states for which the game terminates with $\gamma_f > 0$ (one-on-one with D_1) and $\gamma_f < 0$ (one-on-one with D_2), respectively (c.f. Lemma 8). Similarly, we define the region $\mathscr{R}_{d_{1,2}}$ as the set of states for which the game terminates with $\gamma_f = 0$ which is completely specified by Lemma 7 and Lemmas 10–10.

Theorem 4. For states in the region $\mathscr{R}_{d_{1,2}}$ the equilibrium Attacker heading angle is

$$\psi^* = \gamma + \sin^{-1}\left(\frac{R\sin\gamma}{\nu\alpha}\right) \tag{58}$$

and the Value function is

$$V(\mathbf{x}) = -R_f = \nu\alpha\frac{\sin\psi^*}{\sin\gamma}. \tag{59}$$

Proof. The result follows from Lemmas 9–11 via a geometric proof process very similar to Theorem 3; the details are omitted. $\square$

C. Full Solution

The two Defender game is truly three dimensional (in the reduced state space, i.e., R, γ, α). Although one may obtain the equilibrium flowfield over the whole state space by substituting the equilibrium strategies into the kinematics, it is more illustrative to visualize the solution in the (x, y)-plane for a particular α. Figure 4 shows the full solution of the two-Defender one-Attacker game, including all of the separating surfaces, regions, and salient features along with several representative Attacker trajectories.

IV. Conclusion

The problem of guarding a circular target by patrolling its perimeter was considered. We formulated the one-Defender one-Attacker and two-Defender one-Attacker scenarios as zero-sum differential games with different cost/payoff functionals depending on whether the Attacker could reach the target's perimeter before the Defender(s) could 'lock on'. The analysis formally verifies that the Attacker heading strategy given in the literature for the one-Defender scenario is indeed the saddle-point equilibrium strategy for the games posed here [3]. For the two-Defender scenario, the state space was partitioned into regions based on the equilibrium termination condition. Analytic expressions for the separating surfaces between these regions and Value functions for each case were derived. The Attacker strategy in the Defenders-win, symmetric termination region differs from that of [3], partly due to differences in the termination condition and cost/payoff functional.

Future work on this problem may include investigating different cost/payoff functionals, such as $\min\max$ time for the Attacker to reach the target. Additionally, understanding the impact of these termination conditions on multi-Attacker multi-Defender scenarios is of interest.

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References

[1] R. Isaacs, *Differential Games: A Mathematical Theory with Applications to Optimization, Control and Warfare*. Wiley, New York, 1965.

[2] M. Pachter, E. Garcia, and D. W. Casbeer, "Active target defense differential game," in *2014 52nd Annual Allerton Conference on Communication, Control, and Computing*. IEEE, 9 2014. [Online]. Available: http://dx.doi.org/10.1109/allerton.2014.7028434

[3] D. Shishika and V. Kumar, "Local-game decomposition for multiplayer perimeter-defense problem," in *2018 IEEE Conference on Decision and Control (CDC)*. IEEE, 12 2018. [Online]. Available: http://dx.doi.org/10.1109/cdc.2018.8618879

[4] ——, "Perimeter-defense game on arbitrary convex shapes," *arXiv*, 2019. [Online]. Available: https://arxiv.org/abs/1909.03989

[5] D. Shishika, J. Paulos, and V. Kumar, "Cooperative team strategies for multi-player perimeter-defense games," *Robotics and Automation Letters*, pp. 2738–2745, 2020.

[6] R. Isaacs, "Differential games: Their scope, nature, and future," *Journal of Optimization Theory and Applications*, vol. 3, pp. 283–295, 5 1969. [Online]. Available: https://doi.org/10.1007/BF00931368

[7] J. V. Breakwell, *Zero-sum differential games with terminal payoff*, ser. Lecture Notes in Control and Information Sciences. Springer-Verlag New York/Berlin, 1977, vol. 3, pp. 70–95.

[8] T. Başar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, 2nd ed., ser. Mathematics in Science and Engineering. Elsevier, 1982, vol. 160. [Online]. Available: https://www-sciencedirect-com.wrs.idm.oclc.org/science/journal/00765392/160/supp/C

[9] A. E. Bryson and Y.-C. Ho, *Applied Optimal Control: Optimization, Estimation and Control*. CRC Press, 1975.

[10] Z. Akilan and Z. Fuchs, "Zero-sum turret defense differential game with singular surfaces," in *2017 IEEE Conference on Control Technology and Applications (CCTA)*, 2017, pp. 2041–2048.