TURRET ESCAPE DIFFERENTIAL GAME

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Abstract. In this paper, a zero-sum differential game is formulated and solved in which a mobile Evader seeks to escape from within a circle at whose origin lies a stationary, turn-constrained Turret. The scenario is a variant of the famous Lady in the Lake game in which the shore-constrained Pursuer has been replaced with the Turret. As in the former, it is assumed that the Turret's maximum angular rate is greater than the linear velocity of the Evader. Since two outcomes are possible, a Game of Kind arises - either the Evader wins by reaching the perimeter of the circle, or the Turret wins by aligning with the latter's position. A barrier surface partitions the state space into two regions corresponding to these two outcomes and a Game of Degree is solved within each region. The solutions to the Games of Degree are comprised of the Value functions (i.e., the equilibrium value of the cost/utility as a function of the state) and the saddle-point equilibrium control policies for the two players. Like the Lady in the Lake game, the equilibrium policy of the Evader is not uniquely defined where it has angular rate advantage over the Turret. Unlike the Lady in the Lake game, the losing region for the Evader is present for all speed ratios, and there is an additional semi-permeable surface separating center- and shore-bound Evader trajectories. The solution depends heavily upon the speed ratio of the agents; in particular, there are two speed ratio regimes with distinctive solution structures.

1. Introduction

Evasion from ground-based defensive sites is an important scenario to consider for aerial vehicles. This paper is concerned with an aerial vehicle who, while completing its mission, discovers that it is in firing range of a turn-constrained Turret. The aerial vehicle, or Evader, is not equipped to destroy the Turret and must instead attempt to escape. If the Evader can make it outside of the range of the Turret (which is assumed to be known by the Evader), then the Evader is considered to be safe. Otherwise, if the Turret can align its line-of-sight with the Evader's position, then the Evader is assumed to be neutralized.

In order to formalize the scenario and address questions such as 'Can the Evader escape?', 'With what margin?', etc., we employ the theory of differential games [1]. In particular, we formulate a two-person zero-sum differential game, a *Game of Degree*, for both the Evader- and Turret-winning cases. Additionally, the manifold

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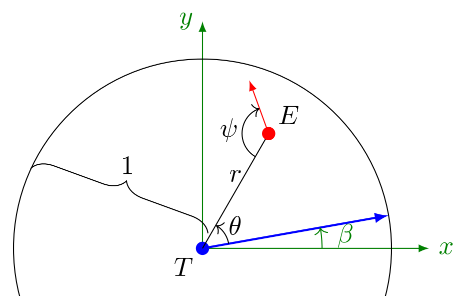


FIGURE 1. Schematic representation of the scenario with salient states identified. The global Cartesian coordinate system is shown in green.

which delineates Evader- and Turret-win, the *Game of Kind* surface, is obtained via the solution of these games.

In lieu of a Turret, one may consider the turn-constrained agent to be some kind of sensor with limited range and the Evader simply seeks to avoid detection (c.f., [2], [3]). The scenario described above is also related to the so-called High Value Target scenario [4] where an Intruder approaches a target, then it may see that it will lose to the Defender, and subsequently must escape from the Defender.

This scenario is also related to the *Lady in the Lake* problem. Although the problem itself was posed and discussed earlier, we will utilize [5] to refer to the problem and some details concerning its solution. There, a faster Pursuer is constrained to not be able to enter the circular region, and the Evader seeks to maximize angular separation when she reaches the circle's perimeter from inside. The biggest difference in the Turret formulation of the problem is that the scenario terminates if $\theta = 0$. In the Lady in the Lake problem, the Evader is safe at any point inside the lake, and, more importantly, can *always* maneuver back inside the ν-circle and "restart" the engagement.

There have been many recent papers concerning differential games involving a Turret and mobile agent. In [6], the mobile agent is an Attacker who seeks to collide with the Turret whilst avoiding its line-of-sight. Because the cost functional is integral and dependent on time and relative look-angle over the equilibrium Attacker trajectories are curved in the global Cartesian frame. Most notably, [7], [8] solves a similar game but the scenario terminates in the same way as the Turret Escape Differential Game (i.e., with Attacker reaching the circle containing the Turret or with the Turret aligning with the Attacker). There, as in [6], the Attacker begins *outside* the circle containing the Turret. Lastly, [9] extends [7] by considering the possibility of the Attacker choosing to retreat to some safe zone in lieu of engaging the Turret.

The kinematics of the system under consideration are

$$f\left(\mathbf{z}, u_T, \psi\right) = \dot{\mathbf{z}} = \begin{bmatrix} \dot{r} \\ \dot{\theta} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -\nu\cos\psi \\ \frac{\nu}{r}\sin\psi - u_T \\ u_T \end{bmatrix}, \tag{1.1}$$

where $\nu < 1$ and $u_T \in [-1, 1]$. Note these are the nondimensionalized kinematics which are normalized such that the lake's radius is 1, T's angular speed is 1, and $\nu < 1$ is the ratio of E's linear speed to T's angular speed (c.f. [7] for details). Fig. 1 shows a schematic representation of the scenario.

The goal of this investigation is to determine 1) the region of the state space in which E can be guaranteed to escape before T's look-angle is aligned with its position, and 2) a control strategy for E to ensure escape. Both the escape region and an admissible escape strategy may be obtained by formulating a zero-sum differential game with terminal conditions and cost functional defined in the following sections.

It is assumed that the initial conditions are such that $0 < r < 1$ and $\theta \neq 0$, thus let the space over which the game is played be defined

$$\Omega \equiv \{\mathbf{z} \mid 0 < r < 1,\ \theta \neq 0\}. \tag{1.2}$$

Then this space is partitioned into two regions, $\mathscr{R}_E$ and $\mathscr{R}_T$, the Evader-winning and Turret-winning regions, respectively, such that $\Omega = \mathscr{R}_E \cup \mathscr{R}_T$ and $\mathscr{R}_E \cap \mathscr{R}_T = \emptyset$. The surface which lies on the boundary of these two regions is called the *Game of Kind* surface, which is denoted as $\mathscr{K}$. It is assumed that the point where $(r, \theta) = (1, 0)$, which is akin to a tie, is in $\mathscr{R}_E$.

The remainder of the paper is organized as follows. Section 2 covers the *Game of Degree* in which the Evader wins by escaping the Turret's range. Section 3, likewise, covers the *Game of Degree* in which the Turret wins by neutralizing the Evader before it can get out of range. These two sections establish what we refer to as the regular strategy/solution in which the Evader actively maneuvers away from the Turret. Then, Section 4 pieces the complete solution together, taking into consideration the possibility of the Evader entering the region of angular speed advantage as well as two important singularities. Lastly, the paper is concluded in Section 5.

2. Game with Evader Winning

2.1. **Analysis.** Let the terminal boundary condition be written

$$\phi(\mathbf{z}_f) = r_f - 1. \tag{2.1}$$

Then the terminal surface is defined as

$$\mathcal{T} \equiv \{\mathbf{z} \mid \phi = 0\}. \tag{2.2}$$

It has been assumed that escape is possible; we proceed therefore with a cost functional based on the terminal angular separation angle:

$$J(\mathbf{z}; u_T(\cdot), \psi(\cdot)) = \Phi(\mathbf{z}_f) = |\theta_f|. \tag{2.3}$$

Note this cost functional is of Mayer type (i.e., a function only of the terminal state and/or time).

The Value function is then the min max (or max min) value of the cost functional, (2.3), representing the saddle-point equilibrium value of the two-player zero-sum differential game:

$$\begin{aligned} V(\mathbf{z}_0) &= \min_{u_T(\cdot)} \max_{\psi(\cdot)} J(\mathbf{z}_0; u_T(\cdot), \psi(\cdot)) \\ &= \min_{u_T(\cdot)} \max_{\psi(\cdot)} |\theta_f|. \end{aligned} \tag{2.4}$$

Now, the first order necessary conditions for equilibrium will be utilized to characterize the equilibrium control inputs. Let $\boldsymbol{\lambda} \equiv \begin{bmatrix}\lambda_r & \lambda_\theta & \lambda_\beta\end{bmatrix}^\top$ be a vector of adjoint variables. The Hamiltonian of the system is

$$\mathscr{H}\left(\mathbf{z}, \boldsymbol{\lambda}, t\right) = \dot{\mathbf{z}} \cdot \boldsymbol{\lambda} = -\lambda_r \nu \cos\psi + \lambda_\theta \left(\frac{\nu}{r}\sin\psi - u_T\right) + \lambda_\beta u_T. \tag{2.5}$$

The equilibrium adjoint dynamics are given by [10]

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial \mathscr{H}}{\partial \mathbf{z}} = \begin{bmatrix}\lambda_\theta \frac{\nu}{r^2}\sin\psi & 0 & 0\end{bmatrix}^\top \tag{2.6}$$

The transversality condition [10] gives the value of the adjoint variables at final time

$$\boldsymbol{\lambda}_f^\top = \frac{\partial \Phi}{\partial \mathbf{z}_f} + \mu \frac{\partial \phi}{\partial \mathbf{z}_f}, \tag{2.7}$$

where μ is another adjoint variable whose value is constant. Substituting (2.3) and (2.1) into (2.7) gives

$$\boldsymbol{\lambda}_f^\top = \begin{bmatrix}\lambda_{r_f} & \lambda_{\theta_f} & \lambda_{\beta_f}\end{bmatrix} = \begin{bmatrix}0 & \pm 1 & 0\end{bmatrix} + \mu \begin{bmatrix}1 & 0 & 0\end{bmatrix} = \begin{bmatrix}\mu & \pm 1 & 0\end{bmatrix}, \tag{2.8}$$

where $\operatorname{sign}\left(\lambda_{\theta_f}\right) = \operatorname{sign}\left(\theta_f\right)$. Since $\lambda_{\beta_f} = 0$ and $\dot{\lambda}_\beta = 0$, we have that $\lambda_\beta(t) = 0$ for all $t \in [0, t_f]$.

The value of the Hamiltonian at final time is given by [10]

$$\mathscr{H}\left(\mathbf{z}_f, \boldsymbol{\lambda}_f, t_f\right) = \mathscr{H}_f = -\frac{\partial \Phi}{\partial t_f} - \mu \frac{\partial \phi}{\partial t_f} = 0. \tag{2.9}$$

The system, (1.1), is time-autonomous and thus $\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}t} = 0$ which implies that $\mathscr{H} = 0$ for all $t \in [0, t_f]$.

The equilibrium controls, u_T^* and ψ^*, must minimize (and maximize) the Hamiltonian, respectively,

$$\begin{aligned} u_T^* &= \arg\min_{u_T(\cdot)} \mathscr{H} = \operatorname{sign}(\lambda_\theta) = \operatorname{sign}(\theta_f) \\ \psi^* &= \arg\max_{\psi(\cdot)} \mathscr{H} \implies \qquad \cos\psi^* = \frac{-\lambda_r}{\sqrt{\lambda_r^2 + \frac{\lambda_\theta^2}{r^2}}}, \qquad \sin\psi^* = \frac{\lambda_\theta}{r\sqrt{\lambda_r^2 + \frac{\lambda_\theta^2}{r^2}}} \end{aligned} \tag{2.10}$$

Substituting the terminal adjoint values, (2.8), and equilibrium controls, (2.10), into (2.9) gives

$$\mathscr{H}_f = \nu \sqrt{\mu^2 + \frac{1}{r_f^2}} - 1 = 0. \tag{2.11}$$

Substituting the terminal value of $r_f = 1$ in and solving for the adjoint variable μ gives $\mu = \pm\sqrt{\frac{1}{\nu^2} - 1}$. We are interested in trajectories which terminate on the $r = 1$ circle *from within* the circle. It must be the case that $\cos\psi_f^* < 0$, and $\cos\psi_f^* \propto -\mu$ which implies

$$\mu = +\sqrt{\frac{1}{\nu^2} - 1}. \tag{2.12}$$

In a similar way, the quantity λ_r, at generic time t, can be obtained.

$$\lambda_r = \pm\sqrt{\frac{1}{\nu^2} - \frac{1}{r^2}}. \tag{2.13}$$

Note the sign of λ_r is not known directly because $\lambda_{r_f} = \mu > 0$ and $\dot{\lambda}_r > 0 \forall t$ according to (2.6), and thus it may have been possible that λ_r crossed 0 on its way to $\mu > 0$.

Lemma 2.1. *The regular equilibrium control strategies for the Turret and Evader for the game of* $\min_{u_T} \max_{\psi} |\theta_f|$ *are*

$$u_T^* = \operatorname{sign}(\theta) \tag{2.14}$$

$$\cos\psi_{reg} \equiv -\sqrt{1 - \frac{\nu^2}{r^2}}, \qquad \sin\psi_{reg} \equiv \operatorname{sign}(\theta)\frac{\nu}{r}. \tag{2.15}$$

respectively, for $r > \nu$*.*

Proof. First, for T's control, (2.14) is obtained by replacing $\operatorname{sign}(\theta_f)$ in the general form, (2.10), with $\operatorname{sign}(\theta)$. This can be done because θ cannot change sign during equilibrium play (when $r \geq \nu$) because doing so would require crossing over $\theta = 0$ (which corresponds to neutralization, in this case) or $\theta = \pi$. The Evader cannot force the system across $\theta = \pi$ when $r \geq \nu$ because it does not have an angular rate advantage over T.

For E's control, substituting (2.13) into (2.10) gives

$$\cos\psi^* = \pm\sqrt{1 - \frac{\nu^2}{r^2}}, \qquad \sin\psi^* = \operatorname{sign}(\theta)\frac{\nu}{r} \tag{2.16}$$

Let us designate the outward-bound version of (2.16), in which the $\cos\psi^*$ term is specialized to be negative, as the regular strategy (i.e., (2.15)) as this is the heading for which E is actively escaping. ☐

Remark 2.2. Just as in [7], [11], the equilibrium Evader trajectory is a straight line in the Cartesian frame (c.f., [7, Lemma 2]). Additionally, the $\sin\psi^*$ component of the Evader's heading is identical to the referenced works. In *this* version of the scenario, however, the Evader's $\cos\psi^*$ component may be pointed towards the origin, initially, and eventually pointing away rather than always towards as in [7].

2.2. Regular Equilibrium Flowfield.

Here, we examine what we refer to as the regular equilibrium dynamics corresponding to Evader trajectories aimed *away* from the ν-circle tangent point. This is in contrast to non-regular trajectories in which E first maneuvers into the ν-circle. Thus, we set $\psi = \psi_{\text{reg}}$. For convenience, the * superscript will be dropped in the following notation.

Lemma 2.3. *The regular equilibrium flowfield for the Evader-winning game is*

$$\theta(r; r_f, \theta_f) = \operatorname{sign}(\theta_f)\left(\sqrt{\frac{r_f^2}{\nu^2} - 1} + \sin^{-1}\left(\frac{\nu}{r_f}\right) - \sqrt{\frac{r^2}{\nu^2} - 1} - \sin^{-1}\left(\frac{\nu}{r}\right)\right) + \theta_f \tag{2.17}$$

Proof. Substituting (2.14) and (2.15) into (1.1) gives

$$f(\mathbf{z}, u_T, \psi) = \begin{bmatrix} \dot{r} \\ \dot{\theta} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \nu\sqrt{1 - \frac{\nu^2}{r^2}} \\ \operatorname{sign}(\theta)\left(\frac{\nu^2}{r^2} - 1\right) \\ \operatorname{sign}(\theta) \end{bmatrix} \tag{2.18}$$

Since $r(t)$ and $\theta(t)$ are monotonic we can write

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\dot{\theta}}{\dot{r}} = \frac{\mathrm{sign}(\theta)\left(\frac{\nu^2}{r^2} - 1\right)}{\nu\sqrt{1 - \frac{\nu^2}{r^2}}} \tag{2.19}$$

$$\implies -\nu \int_{\theta}^{\theta_f} \mathrm{d}\theta = \mathrm{sign}(\theta) \int_{r}^{r_f} \sqrt{1 - \frac{\nu^2}{r^2}}\, \mathrm{d}r \tag{2.20}$$

$$\begin{aligned} -\nu\left(\theta_f - \theta\right) = \ & \mathrm{sign}(\theta)\left(\sqrt{r_f^2 - \nu^2} + \nu \sin^{-1}\left(\frac{\nu}{r_f}\right)\right. \\ & \left. - \sqrt{r^2 - \nu^2} - \nu \sin^{-1}\left(\frac{\nu}{r}\right)\right) \end{aligned} \tag{2.21}$$

which simplifies to (2.17). □

Remark 2.4. This equilibrium flowfield is nearly identical to [7] except that it is negative and we have $\nu \leq r \leq r_f$.

Now we obtain the first piece of the *Game of Kind* surface (which partitions Ω into $\mathscr{R}_E$ and $\mathscr{R}_T$). This piece corresponds to the locus of positions in which E reaches $r = 1$ exactly when $\theta \to 0$, which is obtained by setting $r_f = 1$ and $\theta_f = 0$ in (2.17):

$$\theta_{GoK_1}(r) = \pm\left(\sqrt{\frac{1}{\nu^2} - 1} + \sin^{-1}(\nu) - \sqrt{\frac{r^2}{\nu^2} - 1} - \sin^{-1}\left(\frac{\nu}{r}\right)\right), \tag{2.22}$$

where $\nu < r < 1$. Therefore, the region in which E can safely reach $r = 1$ under the regular strategy is

$$\mathscr{R}_1 = \left\{\mathbf{z} \mid r > \nu \text{ and } \theta \geq \theta_{GoK_1}(r)\right\}. \tag{2.23}$$

Based on the definition in (2.4), the Value function associated with regular trajectories can now be written as

$$V_{\mathrm{reg}}(\mathbf{z}) = |\theta| - \sqrt{\frac{1}{\nu^2} - 1} - \sin^{-1}\nu + \sqrt{\frac{r^2}{\nu^2} - 1} + \sin^{-1}\left(\frac{\nu}{r}\right), \tag{2.24}$$

where $\mathbf{z} \in \mathscr{R}_1$.

Figure 2 shows an example of the region created by θ_{GoK_1}. Note that the region contains initial E positions for which E would be able to reach the ν-circle successfully if it did not implement (2.15).

As mentioned earlier, when E starts in the region of angular velocity advantage (i.e., $r \leq \nu$), it is best for E to maneuver to $(r, \theta) = (\nu, \pi)$ to maximize $|\theta_f|$. The corresponding trajectories are found by setting $(r, \theta) = (\nu, \pi)$ and $r_f = 1$ in (2.17) and solving for θ_f:

$$\theta_f\big|_{r=\nu,\theta=\pi,r_f=1} = \pm\left(\pi - \left(\sqrt{\frac{1}{\nu^2} - 1} + \sin^{-1}(\nu) - \frac{\pi}{2}\right)\right) \tag{2.25}$$

Then, the curve $\theta(r; 1, \theta_f)$ with θ_f given above and $r \in [\nu, 1]$ can be computed; Fig. 3 shows an example of these trajectories.

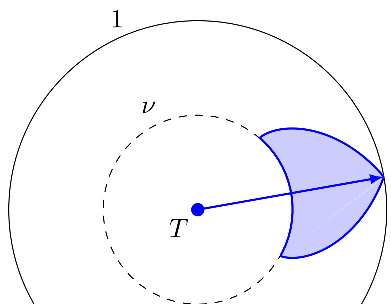


FIGURE 2. Evader lose region for $\nu = 0.5$ under regular Evader strategy (2.15). This region also corresponds to $\mathscr{R}_1^c$.

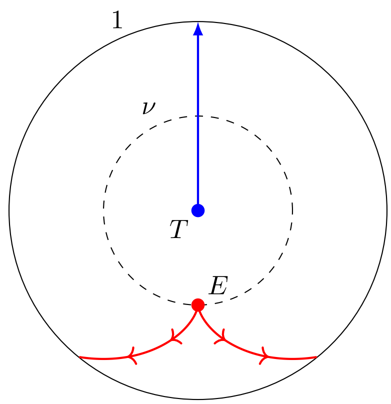


FIGURE 3. Equilibrium Evader trajectories emanating from $(r, \theta) = (\nu, \pi)$. The Value of these trajectories is $\theta_f = 141°$ for $\nu = 0.5$. This figure is analogous to [5, Fig. 14].

Lemma 2.5. *The point $(r, \theta) = (\nu, \pi)$ is in the Evader's win region, $\mathscr{R}_E$ if and only if $\nu \geq \nu_{crit}$ where*

$$\nu_{crit} \approxeq 0.21723 \tag{2.26}$$

Proof. The critical case for E being able to be able to escape from the point $(r, \theta) = (\nu, \pi)$ is when $\theta_f = 0$ in (2.25) which yields the minimum speed ratio given above. $\square$

Note that this result is identical to the *Lady in the Lake* problem [5, p.370].

3. GAME WITH TURRET WINNING

Now, we specify a new differential game which takes place in the Turret's winning region. For convenience, the notation used in the previous section will be reused and redefined. In $\mathscr{R}_T$, the Turret can guarantee to be able to neutralize the Evader. Thus the terminal boundary condition is

$$\phi(\mathbf{z}_f) = |\theta_f|, \tag{3.1}$$

and the associated terminal surface is, once again, given by (2.2) (i.e., $\theta_f = 0$).

A natural cost functional to consider in the case that E will be neutralized is distance: E should try to get as far from T as it can. Thus, let the cost functional be

$$J\left(\mathbf{z}; u_T(\cdot), \psi(\cdot)\right) = \Phi\left(\mathbf{z}_f\right) = r_f. \tag{3.2}$$

As before, this cost functional is of Mayer type. The Value function is then defined as

$$V(\mathbf{z}_0) = \min_{u_T(\cdot)} \max_{\psi(\cdot)} J\left(\mathbf{z}_0; u_T(\cdot), \psi(\cdot)\right) = \min_{u_T(\cdot)} \max_{\psi(\cdot)} \; r_f. \tag{3.3}$$

The analysis then proceeds in much the same way as in Section 2. Thus, only the significant differences will be highlighted. For example, the Hamiltonian and equilibrium adjoint dynamics are, again, given by (2.5) and (2.6), respectively. However, substituting (3.1) into (2.7) gives

$$\boldsymbol{\lambda}_f^\top = \begin{bmatrix} 1 & \pm\mu & 0 \end{bmatrix}, \tag{3.4}$$

with $\mu > 0$, which is, essentially, a scaled version of (2.8). Substituting (3.4) and (2.10) into (2.9) gives

$$\mathscr{H}_f = \nu\sqrt{1 + \frac{\mu}{r_f^2}} - \mu = 0. \tag{3.5}$$

At this point, r_f is unknown, but the above expression can be rearranged to obtain μ in terms of r_f, giving

$$\operatorname{sign}\left(\theta_f\right)\lambda_\theta = \mu = \tfrac{\nu r_f}{\sqrt{r_f^2 - \nu^2}}. \tag{3.6}$$

As was done in the previous section, the focus is on regular trajectories wherein E is heading away from T. This implies that $\lambda_r > 0$.

Lemma 3.1. *The regular equilibrium control strategies for the Turret and Evader for the game of* $\min_{u_T} \max_\psi r_f$ *are*

$$\begin{gathered} u_T^* = \operatorname{sign}(\theta) \\ \cos\psi_{reg} = -\sqrt{1 - \frac{\nu^2}{r^2}}, \qquad \sin\psi_{reg} = \operatorname{sign}(\theta)\,\frac{\nu}{r}. \end{gathered} \tag{3.7}$$

Proof. Substituting (3.6) into the equilibrium Hamiltonian at general time and rearranging for λ_r gives

$$\lambda_r = \frac{r_f}{r}\sqrt{\frac{r^2 - \nu^2}{r_f^2 - \nu^2}}. \tag{3.8}$$

Finally, substituting into the general equilibrium control expressions, (2.10), yields the regular equilibrium control strategies in (3.7). $\square$

Corollary 3.2. *The equilibrium flowfield for the Turret-winning game is given by* (2.17).

Proof. The regular equilibrium control strategies for the Turret-winning game, (3.7), are identical to those for the Evader-winning game (c.f. (2.14) and (2.15)). Therefore, the Turret-winning regular equilibrium flowfield is identical to the Evader-winning regular equilibrium flowfield derived in Section 2, (2.17). □

Finally, the regular Value of the game with the Turret winning is the solution of the transcendental equation (for V_{reg})

$$0 = \sqrt{\frac{V_{\text{reg}}^2}{\nu^2} - 1} + \sin^{-1}\left(\frac{\nu}{V_{\text{reg}}}\right) - \sqrt{\frac{r^2}{\nu^2} - 1} - \sin^{-1}\left(\frac{\nu}{r}\right) - |\theta|, \tag{3.9}$$

for $\mathbf{z} \notin \mathscr{R}_1, r > \nu$, which is obtained from (2.17) by recalling $V \equiv r_f$ for this game and $\theta_f = 0$, by construction.

4. Solution Construction

Up to now, it has been hinted that, depending on the Evader's position and the speed ratio, it may be advantageous for E to first enter the circle of radius ν prior to heading away from T. In this section, this aspect is addressed along with some singularities which are present in both *Games of Degree*.

The state space is such that $0 < r < 1$. When $r < \nu$, the Evader has an angular velocity advantage and can therefore safely arrive at a position $(r, \theta) = (\nu, \pi)$ in a myriad of ways. From there, based on the regular strategy (which is the same for both *Games of Degree*, c.f., (2.15) and (3.7)), E would exit the circle of radius ν tangentially (in a direction corresponding to T's instantaneous choice). Doing so has some associated Value, defined as $V_\nu \equiv V_{\text{reg}}(r = \nu, \theta = \pi)$ for both the Evader-winning and Turret-winning games. Then, E must compare V_ν with the Value associated with the regular strategy for the appropriate game to determine whether it is advantageous to enter the ν-circle, reach (ν, π), and subsequently head away from T, or immediately head away from T. Thus the overall structure of the Turret Escape Differential Game hinges on whether the point $(r, \theta) = (\nu, \pi)$ is in the Evader- or Turret-winning region. In fact, by the definition of ν_{crit} (obtained by setting (2.25) to zero and solving for ν), if $\nu < \nu_{\text{crit}}$ then $(\nu, \pi) \in \mathscr{R}_T$, otherwise $(\nu, \pi) \in \mathscr{R}_E$.

Let $\mathscr{R}_\nu$ be defined as the region in which E can reach $r \leq \nu$ safely. Note that $\mathscr{R}_\nu$ includes all of $r \leq \nu$ by definition. Then the Evader's winning region is given by

$$\mathscr{R}_E = \begin{cases} \mathscr{R}_1 \cup \mathscr{R}_\nu & \text{if } \nu > \nu_{\text{crit}}, \\ \mathscr{R}_1 & \text{otherwise} \end{cases} \tag{4.1}$$

and the Turret's winning region is simply $\mathscr{R}_T = \Omega \setminus \mathscr{R}_E$. It remains to construct the region $\mathscr{R}_\nu$ mathematically and assemble the full solution.

4.1. ν-circle Reachability.

This section is concerned with constructing the region for which E can safely reach $r = \nu$ from $\nu < r_0 < 1$. Incidentally, this auxiliary game (in which E and T wish to min and max, respectively, $|\theta|$ at the time when $r = \nu$) is nearly identical to the single-Attacker Turret Defense problem [7] with the Evader, in this case, behaving like the Attacker. The quantities associated with this auxiliary game are denoted with a subscript ν (for "ν-circle reachability").

A simple scaling is needed in order to obtain the auxiliary *Game of Kind* surface. Define the following

$$\hat{r} \equiv \frac{r}{\nu}, \quad \hat{t} \equiv \frac{1}{\nu} t, \tag{4.2}$$

which results in a scaled speed ratio

$$\hat{\nu} = 1. \tag{4.3}$$

Then, from [7], the auxiliary *Game of Kind* surface is given by

$$\theta_{GoK_\nu}(\hat{r}) = \sqrt{\frac{\hat{r}^2}{\hat{\nu}^2} - 1} + \sin^{-1}\left(\frac{\hat{\nu}}{\hat{r}}\right) - \sqrt{\frac{1}{\hat{\nu}^2} - 1} - \sin^{-1}(\hat{\nu}) \tag{4.4}$$

$$\implies \theta_{GoK_\nu}(r) = \sqrt{\frac{r^2}{\nu^2} - 1} + \sin^{-1}\left(\frac{\nu}{r}\right) - \frac{\pi}{2}, \tag{4.5}$$

for $r \in [\nu, 1]$. Therefore, the region in which E can guarantee to reach the ν-circle safely is

$$\mathscr{R}_\nu = \{\mathbf{z} \mid |\theta| > \theta_{GoK_\nu}(r) \text{ or } r < \nu\}. \tag{4.6}$$

The strategy corresponding to E maximizing $|\theta|$ at the time when $r = \nu$ is, again, given by a scaled version of the strategy in [7]:

$$\cos\psi_\nu^* = \sqrt{1 - \frac{\hat{\nu}^2}{\hat{r}^2}} \qquad \sin\psi_\nu^* = \text{sign}(\theta)\,\frac{\hat{\nu}}{\hat{r}} \tag{4.7}$$

$$\implies \cos\psi_\nu^* = \sqrt{1 - \frac{\nu^2}{r^2}} \qquad \sin\psi_\nu^* = \text{sign}(\theta)\,\frac{\nu}{r}. \tag{4.8}$$

As concerns the original cost functionals for the Evader- and Turret-winning games, any admissible Evader control, ψ, is optimal. By admissible, it is meant that E reaches the circle of radius ν without being neutralized by T. The heading ψ_ν^*, given above, is one such heading which is guaranteed to be admissible over the entire ν-circle reachable region, $\mathscr{R}_\nu$. Interestingly, ψ_ν^* satisfies the first order necessary conditions for optimality with respect to both of the original *Games of Degree* (i.e., it is the positive version of (2.16)).

From Fig. 4 there *is* overlap of the auxiliary win region, $\mathscr{R}_\nu$, with both the Evader win and lose regions of the game of interest under the Evader strategy in (2.15) ($\mathscr{R}_1$ and $\mathscr{R}_1^c$, respectively). Clearly, if $\mathbf{z} \in \mathscr{R}_\nu \cap \mathscr{R}_1^c$ (i.e., E can reach the ν-circle and would lose under (2.15)) then E should enter the ν-circle whereby it can reach $(r, \theta) = (\nu, \pi)$ and win the game with a Value of 141° (for $\nu = 0.5$).

4.2. **Turret Dispersal Surface.** As is typical in games involving Turrets (c.f., [6]–[9]), there is a Dispersal Surface (DS) at $\cos(\theta) = -1$, i.e., when T is facing away from E (for both *Games of Degree*). On the DS, and where $r \geq \nu$, T is free to choose either CW or CCW; meanwhile, E can only obtain the Value of the game by guessing T's choice and taking the same direction. Otherwise, if E chooses the opposite direction, then E must immediately switch directions. Since T has freedom in its choice, we refer to this surface as the Turret Dispersal Surface (TDS); its formal definition is

$$\mathscr{D}_T \equiv \{\mathbf{z} \mid \nu \leq r < 1,\ \cos(\theta) = -1\}. \tag{4.9}$$

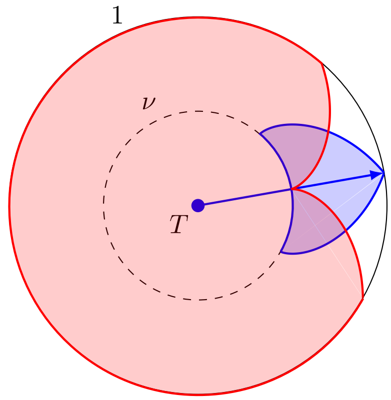


FIGURE 4. Auxiliary Evader win region, $\mathscr{R}_\nu$, (red) wherein E can safely reach the ν-circle superimposed over the Evader lose region under the regular equilibrium strategy (blue).

4.3. **Evader Dispersal Surface.** The Evader Dispersal Surface (EDS) is the manifold of (r,θ) for which E can take ψ_{reg} or ψ_ν^* and achieve the same Value. By construction, this manifold is a subset of the regular equilibrium trajectory emanating from $(r,\theta)=(\nu,\pi)$ (shown, e.g., in Fig. 3) in which E can reach the ν-circle. When $\nu<\nu_{\text{crit}}$, the EDS lies inside the Evader-winning region, $\mathscr{R}_E$, and corresponds to the entire trajectory emanating from (ν,π). When $\nu\geq\nu_{\text{crit}}$ the EDS lies inside the Turret-winning region, $\mathscr{R}_T$ and only a portion of the (ν,π) trajectory lies in $\mathscr{R}_\nu$. At any point along the EDS, E may choose between continuing along the outward trajectory (ψ_{reg}) or turning back to enter the ν-circle and start over (ψ_ν^*). The formal definition of the EDS is

$$\mathscr{D}_E=\left\{\mathbf{z}\mid r\geq\nu,\ (2.17)\text{ with } r_0=\nu,\ \theta_0=\pi,\ \theta\geq\theta_{GoK_\nu}(r)\right\}. \tag{4.10}$$

4.4. **Full Solution.** With the pertinent regions constructed and candidate Values derived, the full solution for each of the *Games of Degree* can be expressed. For the case in which $\nu<\nu_{\text{crit}}$, the Evader cannot win from any point in which $r\leq\nu$. However, if $r\leq\nu$, then E can delay its neutralization indefinitely. We will assume that the in lieu of a draw, E prefers to terminate the game by being neutralized at the farthest distance it can achieve (i.e., the Value corresponding to departing from $(r,\theta)=(\nu,\pi)$, V_ν). For both *Games of Degree* there are two Evader strategies, ψ_{reg} and ψ_ν^* (as shown in Fig. 5), leading to two candidate Values, V_{reg} and V_ν, respectively.

Remark 4.1. For both *Games of Degree*, V_{reg} is only defined in a subset of the space in which $\nu<r<1$, and V_ν is defined only when $\mathbf{z}\in\mathscr{R}_\nu$ (c.f., (4.6)).

For the game with E winning, the Value of the game when E departs from $(r,\theta)=(\nu,\pi)$, V_ν, is given by (2.25), which exists only when $\nu\geq\nu_{\text{crit}}$.

Theorem 4.2. *The solution of the game with Evader winning, i.e.,* $\max_\psi\min_{u_T}|\theta_f|$ *is given by the following Value function*

$$V(\mathbf{z};\nu)=\begin{cases}V_{reg}(\mathbf{z};\nu) & (2.24)\ \textit{if } C_E\\ V_\nu(\nu) & (2.25)\ \textit{otherwise,}\end{cases} \tag{4.11}$$

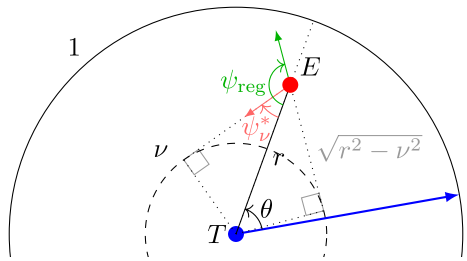


FIGURE 5. The two Evader strategies used in the overall solution for both *Games of Degree*; the colors correspond to the colors in Fig. 6.

for $\mathbf{z} \in \mathscr{R}_E$, *and associated equilibrium Evader control*

$$\psi^*(\mathbf{z};\nu) = \begin{cases} \psi_{reg}(\mathbf{z};\nu) & (2.15)\ \textit{if } C_E \\ \psi_\nu^*(\mathbf{z};\nu) & (4.8)\ \textit{if not } C_E,\ \textit{and } r > \nu \\ \textit{undefined} & \textit{otherwise.} \end{cases} \tag{4.12}$$

where the condition, C_E*, is defined as*

$$C_E \equiv (r > \nu) \wedge ((\nu < \nu_{crit}) \vee (\mathbf{z} \notin \mathscr{R}_\nu) \vee (V_{reg} \geq V_\nu)). \tag{4.13}$$

Proof. Nearly all of the components of the above solution have been proven in the preceding Lemmas and analyses. It remains to prove that the condition C_E is correct.

First, for C_E to be met it must be the case that $r \geq \nu$. If $r < \nu$ then E is inside the ν-circle and thus has advantage over T in terms of angular velocity. For the Evader-winning game, the best possible $|\theta_f|$ that E can achieve in this case is obtained when E starts at the point $(r,\theta) = (\nu,\pi)$ and escapes using the regular strategy, ψ_{reg}. Hence, the Value is V_ν when $r < \nu$. No particular E strategy has been proposed for E to reach (ν,π) from inside the ν-circle, hence ψ^* is undefined when $r < \nu$.

Second, when $r > \nu$, we must consider whether $\nu \lesseqgtr \nu_{\text{crit}}$. If $\nu < \nu_{\text{crit}}$, then E cannot win from the point (ν,π) and there is no need to consider anything but the regular strategy. However, when $\nu \geq \nu_{\text{crit}}$, then we must determine 1) if the ν-circle is reachable, and, if so, whether entering the ν-circle yields a better Value. If either of these last two checks fail, then, again, the regular strategy should be selected. □

For the game with T winning, the Value of the game when E departs from $(r,\theta) = (\nu,\pi)$, V_ν, is given by the solution of the transcendental equation

$$0 = \sqrt{\frac{V_\nu^2}{\nu^2} - 1} + \sin^{-1}\left(\frac{\nu}{V_\nu}\right) - \frac{3\pi}{2}, \tag{4.14}$$

which is obtained by substituting $(r,\theta) = (\nu,\pi)$ into (3.9), and exists only when $\nu < \nu_{\text{crit}}$.

Theorem 4.3. *The solution of the game with Turret winning, i.e.,* $\max_{\psi} \min_{u_T} r_f$ *is given by the following Value function*

$$V(\mathbf{z};\nu) = \begin{cases} V_{reg}(\mathbf{z};\nu) & (3.9) \text{ if } C_T \\ V_{\nu}(\nu) & (4.14) \text{ otherwise,} \end{cases} \tag{4.15}$$

for $\mathbf{z} \in \mathscr{R}_T$*, and associated equilibrium Evader control*

$$\psi^*(\mathbf{z};\nu) = \begin{cases} \psi_{reg}(\mathbf{z};\nu) & (2.15) \text{ if } C_T \\ \psi^*_{\nu}(\mathbf{z};\nu) & (4.8) \text{ if not } C_T, \text{ and } r > \nu \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{4.16}$$

where the condition, C_T*, is defined as*

$$C_T \equiv (r > \nu) \wedge ((\nu \geq \nu_{crit}) \vee (\mathbf{z} \notin \mathscr{R}_{\nu}) \vee (V_{reg} \geq V_{\nu})). \tag{4.17}$$

Proof. The logic of this proof is similar to the proof of the previous Theorem and is omitted. □

Remark 4.4. The control associated with entering the ν-circle, ψ^*_{ν}, is not unique.

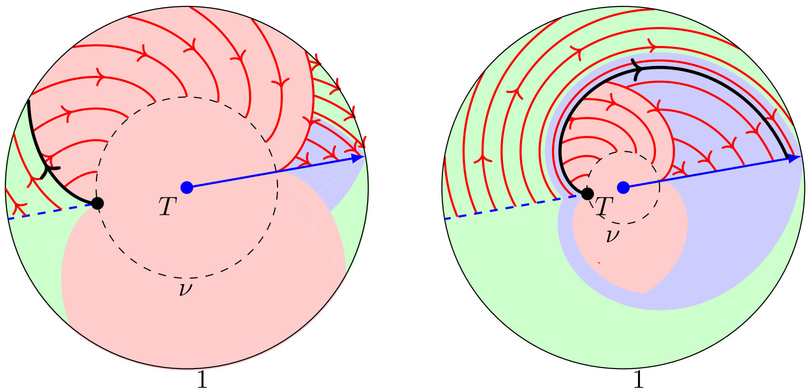
Theorem 4.5. *The Game of Kind surface which partitions* $\mathscr{R}_E$ *and* $\mathscr{R}_T$ *is given by*

$$\mathscr{K} \equiv \begin{cases} \{\mathbf{z} \mid |\theta| = \theta_{GoK_1}(r)\} & \text{if } \nu < \nu_{crit}, \\ \{\mathbf{z} \mid |\theta| = \min\{\theta_{GoK_\nu}(r),\ \theta_{GoK_1}(r)\}\} & \text{otherwise.} \end{cases} \tag{4.18}$$

Proof. When $\nu < \nu_{\text{crit}}$, the point $(r, \theta) = (\nu, \pi)$ is in $\mathscr{R}_T$ due to Lemma 2.5. Therefore, the Evader cannot win by entering the ν-circle, and the only pertinent question is whether E can reach $r = 1$ under the regular strategy which is demarcated by the curve θ_{GoK_1}. When $\nu \geq \nu_{\text{crit}}$, we have $(\nu, \pi) \in \mathscr{R}_E$. So for E to lose, it must be outside $\mathscr{R}_1$ *and* outside $\mathscr{R}_{\nu}$. Therefore, the demarcating curve must be the minimum of θ_{GoK_1} and θ_{GoK_ν}. □

In summary, within the Turret Escape Differential Game there are actually two different *Games of Degree*: a game in which the Evader wins by reaching $r = 1$ (while trying to maximize θ_f), and a game in which the Turret wins by driving $\theta \to 0$ (while E tries to maximize r_f). The full solution is depicted in Fig. 6. Red regions indicate where E heads to the circle of radius ν (the ratio of E's speed to the T's max angular velocity) inside which it has angular rate advantage and is able to reach the solid black point opposite the Turret (i.e., where $(r, \theta) = (\nu, \pi)$). Green regions indicate areas of regular play in which E aims away from the tangent of the ν-circle for the E-winning game. The blue region indicates areas of regular play for the T-winning game. The equilibrium flowfield in the relative coordinate system is indicated by the red trajectories in the portion of the state space where $\theta \in [0, \pi]$. The black trajectory is a semi-permeable surface and also the terminal arc taken by any trajectory leading into the ν-circle. The portion of the black trajectory which borders the red region corresponds to the EDS. This study did not prescribe particular control strategies for states inside the ν-circle (other than to eventually reach (ν, π)), and hence the flowfield only occupies the space where $r > \nu$.

Fig. 7 shows how the size of the Evader's win region changes w.r.t. the speed ratio, ν. Generally, as ν increases, the Evader becomes faster relative to the Turret, and thus its win region grows to cover more of the play area. Note the discontinuity



(A) $\nu = 0.5 > \nu_{\text{crit}}$ (B) $\nu = 0.2 < \nu_{\text{crit}}$

FIGURE 6. State space partitioning and equilibrium flowfield of the Turret Escape Differential Game.

that occurs at $\nu = \nu_{\text{crit}}$ wherein entering the ν-circle becomes a viable strategy for the Evader to win; part of the jump is due to the area inside the ν-circle itself.

5. Conclusion

In this paper, we have formulated and solved a differential game in which an Evader, moving with simple motion, seeks to escape a stationary, turn-constrained Turret by maneuvering beyond the latter's range. Two *Games of Degree* were solved – one which occurs in the Evader's win region, wherein the Evader can guarantee to be able to escape, and one which takes place in the Turret's win region, wherein the Turret can guarantee to be able to neutralize the Evader. The

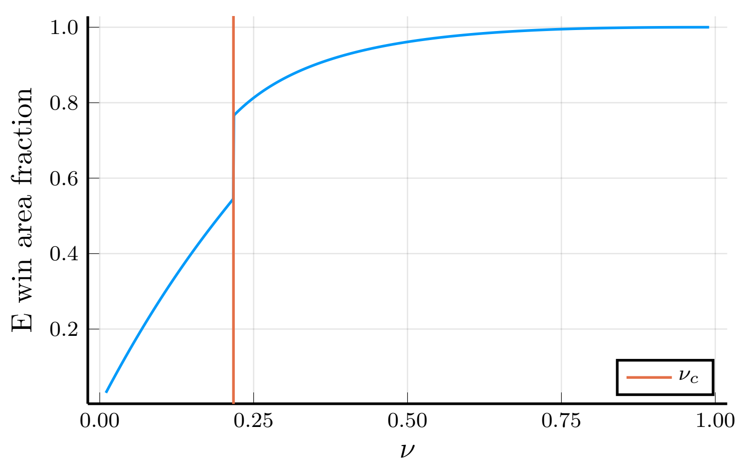


FIGURE 7. The effect of the speed ratio parameter, ν, on the relative size of E's win region (i.e., $|\mathscr{R}_E|/|\Omega|$).

first order necessary conditions for equilibrium were employed to obtain the regular solutions. The regular equilibrium Evader heading is the same for both of these *Games of Degree*. As in the classical *Lady in the Lake* problem, the possibility for the Evader to enter the region of the state space for which it has advantage over the Turret in angular speed is an important feature of the solution. A particular ratio of Evader linear speed to Turret angular speed is important in determining the overall solution structure. When the Evader is fast, entering the region of angular speed advantage is only optimal in a portion of the Evader's win region. When the Evader is slow, entering the region of angular speed advantage is only optimal in a portion of the Turret's win region. The size of the Turret's win region increases as the Evader's speed is decreased.

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