Turret Lock-on in an Engage or Retreat Game

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***Abstract*— In this paper we model and analyze a scenario in the two dimensional plane involving a mobile Attacker and stationary Defender. There are two possibilities for termination: the Attacker can collide with the Defender (engagement) or maneuver to a safe zone away from the Defender (retreat). The Defender is equipped with a directional turret which it can rotate with a bounded rate. If the turret is aligned with the Attacker's position, the Defender has a lock on the Attacker and may choose to fire on the Attacker. Thus, whether engaging or retreating, the Attacker has incentive to evade the turret's line of sight and thereby avoid being locked-on. In the case of retreat, if lock-on occurs the Defender cooperates with the Attacker by withholding fire to allow the Attacker to retreat. However, if the Attacker chooses to engage and lock-on occurs, the Defender will open fire on the Attacker. We model the scenario as a set of differential games with different cost functionals depending on the type of termination. The agents are assumed to have full state information. In the case that the Defender can align with the Attacker the pre- and post-lock portions of the game are solved individually and stitched together. The equilibrium strategies are derived in each case, and a partitioning of the state space wherein a particular termination condition is optimal is constructed.**

I. Introduction

Real-world adversarial scenarios, such as in warfare (e.g. suppression of enemy air defenses), often require decisions to be made based upon complex criteria. Many adversarial scenarios are analyzed under the auspices of differential game theory [1], although, there, the scenario terminates only one way and the agents are concerned with either minimizing or maximizing their respective cost functional – they are not free to deviate from their original goal or intent. We investigate, in this paper, a scenario in which a mobile Attacker, moving with simple motion, is pitted against a static Defender modeled as a turn-constrained turret. The Attacker (denoted A) has a decision to make regarding whether to engage (and destroy) the Defender (denoted D) or retreat to a safe zone. We seek, therefore, not only the equilibrium control strategies for the agents, but also to determine the optimal choice or intention over the whole state space. This is an example of intention (or role) selection, such as whether to be the pursuer or the evader in an aerial dogfight [2]. In the scenario under consideration, D can only pose a threat when its turret's aim is fixed upon the A. Thus there are potentially two segments of the scenario: (1) D seeks to "lock on" to A while the latter has some interest in avoiding the turret's line-of-sight, and (2) after lock, A ultimately proceeds with its intent to engage or retreat. It is possible that lock-on does not occur at all, but we focus, in this paper, on the case in which D can steer its turret onto A's position. Prior to lock-on, A accumulates no cost; afterwards, it has a control- and time-dependent running cost. We refer to the overall scenario as the Lock-Evade, Engage or Retreat (LEER) game. The solution method is based on differential game theory [1], [3], optimal control [4], and the generalized engage or retreat solution [5].

This paper extends the canonical engage or retreat game [6] by considering a turn-constrained defense mechanism. Another turret-based Defender was considered in [7], however, the look-angle dependent portion of the cost functional was smooth (i.e., the turret need not be aimed directly at the Attacker to impose a threat). The discrete switching nature of the cost functional (and thus equilibrium behaviors) places this work in relation with so-called "regime-switching" problems found in the economic differential games literature (see, e.g., [8]). Regime-switching games often exhibit different state evolution (dynamics) when a firm switches regimes. In the case of the LEER game, the evolution of the running cost changes once turret lock-on occurs. Additionally, this work is related to signaling in adversarial confrontations between animals, such as the side-to-side oscillation of honey bees in the presence of predacious hornets [9] to signal a significant retaliation if the hornet enters the nest. Here, D essentially signals a threat to the A by appropriately aiming its turret in order to encourage retreat.

This work combines intent selection [2], [10] with a cost functional that has a state-dependent jump discontinuity. To the authors' knowledge, this work represents a first of its kind. Concerning the LEER problem, itself, we derive the equilibrium Attacker and Defender strategies for engage and retreat up to the time in which D locks onto A. These strategies are used to partition the state space into a region where engaging is optimal and a region where retreating is optimal. Section II provides the problem formulation. The solution methodology is described in Section III, and the derivations for engagement with lock-on and retreat with lock-on are contained in Sections V and VI, respectively. Section VII discusses how to obtain the state space partitioning and provides examples. The paper is concluded in Section VIII.

*The views expressed in this paper are those of the authors and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the United States Government.

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II. Problem Formulation

We begin by defining the state in a relative coordinate system; $\mathbf{z} \equiv [d, \alpha, \beta] \in \mathbb{R}^3$, which is comprised of the distance from A to D, d, the look angle of D w.r.t. A, α, and the azimuth of A's position relative to D w.r.t. the positive x-axis, β. Some of the subsequent analysis is eased by utilizing the Cartesian coordinate system. Let $\tilde{\mathbf{z}} \equiv [x, y, \gamma]$ be the state of the system expressed in the Cartesian frame, where x and y are the coordinates of A and γ is D's look angle w.r.t. the positive x-axis. D is positioned at the origin of the Cartesian coordinate system. The transformation between the two state representations is

$$\begin{bmatrix} x \\ y \\ \gamma \end{bmatrix} = \begin{bmatrix} d\cos\beta \\ d\sin\beta \\ \beta + \alpha \end{bmatrix}. \quad (1)$$

A has a fixed speed (normalized to 1) and controls its instantaneous heading, $\psi \in \mathbb{R}$ (i.e. simple motion, or single integrator dynamics). D is stationary and has control over its turn rate, $\omega \in [-\rho, \rho]$ where $\rho > 1$ is the maximum turn rate. Positive ω corresponds to D turning counterclockwise. D has an additional control variable $w \in [0, \bar{w}]$, which appears in A's cost functional. It represents the amount of turret firepower to apply; the turret must be aimed directly at A for it to be effective. Let A and D's control vectors be defined, respectively, as $\mathbf{u}_A \equiv [\psi]$ and $\mathbf{u}_D \equiv [\omega, w]$. In the relative coordinate system, the kinematics are

$$f(\mathbf{z}, \mathbf{u}_A, \mathbf{u}_D) = \dot{\mathbf{z}} = \begin{bmatrix} \dot{d} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \cos\psi \\ \omega - \frac{1}{d}\sin\psi \\ \frac{1}{d}\sin\psi \end{bmatrix}. \quad (2)$$

The kinematics, expressed in the Cartesian coordinate system, are

$$\tilde{f}(\tilde{\mathbf{z}}, \mathbf{u}_A, \mathbf{u}_D) = \dot{\tilde{\mathbf{z}}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} \cos\tilde{\psi} \\ \sin\tilde{\psi} \\ \omega \end{bmatrix}, \quad (3)$$

where $\tilde{\psi} \equiv \beta + \psi$ is A's heading w.r.t. the positive x-axis. The agents are assumed to have full state information.

Ultimately, A chooses between two endings for the overall LEER scenario: engagement or retreat. In the former, A moves towards D and ultimately collides with it. Let the terminal surface for engagement be defined as

$$\mathcal{E} \equiv \{\mathbf{z} \mid d = 1\} \quad (4)$$

Alternatively, A bypasses D and maneuvers towards a retreat zone, which is specified *a priori*. Here, the retreat zone is the surface $y = y_R$, and thus the terminal surface for retreat is defined as

$$\mathcal{R} \equiv \{\mathbf{z} \mid d\sin\beta = y_R\} \quad (5)$$

The region of admissible initial conditions is defined as

$$\Omega \equiv \{\mathbf{z} \mid d\sin\beta > y_R,\ d > 1\}. \quad (6)$$

Figure 1 shows a diagram of the LEER scenario.

Within both the engage and retreat cases, D may or may not achieve a lock on A. Lock occurs when D's look angle is aligned with A's position (i.e., when $\cos\alpha = 1$). Note that because $\rho > 1$ and $d > 1$ D has an angular velocity advantage over all of Ω; once a lock is achieved, D has sufficient control authority to keep $\cos\alpha = 1$. Thus there are four cases: engagement wherein D achieves a lock on A at some point (Locked Engagement, LE), retreat with a lock (Locked Retreat, LR), engagement without a lock (Unlocked Engagement, UE), retreat without a lock (Unlocked Retreat, UR).

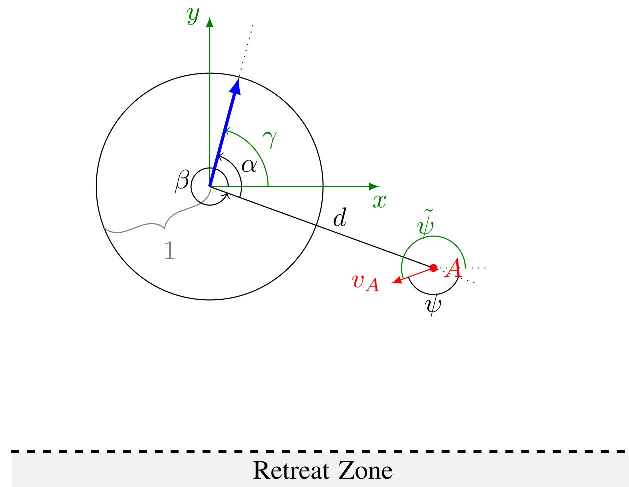

Fig. 1. Lock-Evade, Engage or Retreat Scenario with relative and Cartesian state representations.

The lock function is defined as

$$L(\mathbf{z}) = \begin{cases} 1 & \text{if } \cos\alpha = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

We define cost functionals for A and D, respectively, as

$$J_A(\mathbf{z}_0; \mathbf{u}_A(\cdot), \mathbf{u}_D(\cdot)) = \Psi_A(\mathbf{z}_f) + \int_{t_0}^{t_f} L(\mathbf{z}(t))(w(t) + c)\,\mathrm{d}t \quad (8)$$

$$J_D(\mathbf{z}_0; \mathbf{u}_A(\cdot), \mathbf{u}_D(\cdot)) = \Psi_D(\mathbf{z}_f), \quad (9)$$

where $c > 0$ is a constant time penalty. The terminal cost functions are defined as

$$\Psi_A(\mathbf{z}_f) = \begin{cases} 0 & \mathbf{z}_f \in \mathcal{E} \\ c_A & \mathbf{z}_f \in \mathcal{R} \end{cases} \quad (10)$$

$$\Psi_D(\mathbf{z}_f) = \begin{cases} c_D & \mathbf{z}_f \in \mathcal{E} \\ 0 & \mathbf{z}_f \in \mathcal{R} \end{cases}, \quad (11)$$

where $c_A > 0$ is a constant penalty given to A for retreating instead of engaging, and $c_D > 0$ is a constant penalty given to D if it is destroyed. A and D simultaneously seek to minimize their respective cost functionals, giving rise to a nonzero-sum differential game,

$$J_A^*(\mathbf{z}_0; \mathbf{u}_D(\cdot)) = \min_{\mathbf{u}_A(\cdot)} J_A(\mathbf{z}_0; \mathbf{u}_A(\cdot), \mathbf{u}_D(\cdot)) \quad (12)$$

$$J_D^*(\mathbf{z}_0; \mathbf{u}_A(\cdot)) = \min_{\mathbf{u}_D(\cdot)} J_D(\mathbf{z}_0; \mathbf{u}_A(\cdot), \mathbf{u}_D(\cdot)). \quad (13)$$

D strictly prefers retreat and therefore seeks to make retreat as attractive as possible for A by making engagement as costly as possible. Note that the integral cost in (8) is nonzero only while D has a lock on A. While A can evade D's turret, the integral cost is zero.

For the cases in which D is unable to achieve a lock on A, we have $L = 0$ for all $t \in [t_0, t_f]$ and thus (8) simplifies to $J_A(\mathbf{z}_0; \mathbf{u}_A(\cdot), \mathbf{u}_D(\cdot)) = \Psi_A(\mathbf{z}_f)$. Therefore A's cost is constant: either 0, if $\mathbf{z}_f \in \mathcal{E}$ or c_A, if $\mathbf{z}_f \in \mathcal{R}$. As long as A can guarantee arrival at $\mathcal{E}$ or $\mathcal{R}$ with $\cos\alpha \neq 1$ for all $t \in [t_0, t_f]$ the optimal actions for A and D are not uniquely defined. The construction of trajectories for A which can guarantee satisfaction of this constraint is left for future work.

III. Solution Methodology

In the sections to follow, the two cases wherein D achieves a lock on A, Locked Engagement and Locked Retreat, are analyzed individually. Here, we describe the general solution approach which will be specialized for each of these cases. We will solve the LE and LR cases by splitting the game into a pre-lock and post-lock segment, solving each segment individually, and stitching the solutions together.

For the cases in which D achieves a lock on A, we define the first time at which $\cos\alpha = 1$ as $t_l \in (t_0, t_f]$. Thus the lock function $L = 0$ for $t \in [t_0, t_l)$ and $L = 1$ for $t \in [t_l, t_f]$. The pre-lock segment ends at t_l when $\cos\alpha = 1$; at this point, neither agent has accrued any cost (c.f. (8) and (9)) For the pre-lock segment, we define a terminal-valued zero-sum differential game [1] with a terminal value, $\Phi(\mathbf{z}_l)$, based on the resulting Value of the post-lock segment initialized at $\mathbf{z}_l \equiv \mathbf{z}(t_l)$,

$$\Phi_l(\mathbf{z}_l) = V_{EoR}(\mathbf{z}_l) \tag{14}$$

with A as the minimizer and D as the maximizer, along with the terminal boundary condition

$$\phi_l(\mathbf{z}_l, t_l) = \alpha_l = 0. \tag{15}$$

The Value function for the post-lock segment, V_{EoR}, corresponds to the canonical engage or retreat game [6] and will be defined precisely in Section IV (c.f. (35)). If an equilibrium exists, the Value function of the pre-lock differential game satisfies

$$V_l(\mathbf{z}) = \min_{\mathbf{u}_A(\cdot)} \max_{\mathbf{u}_D(\cdot)} \Phi_l(\mathbf{z}_l, t_l). \tag{16}$$

Because the Value of the pre-lock segment depends on the Value of the post-lock segment, Eqs. (14)–(16) may be thought of as a one-step dynamic programming problem.

The solution methodology proceeds with the formation of the Hamiltonian

$$\mathscr{H} = \dot{\mathbf{z}} \cdot \boldsymbol{\lambda} = \lambda_d \cos\psi + \lambda_\alpha \left(\omega - \frac{1}{d}\sin\psi\right) + \lambda_\beta \frac{1}{d}\sin\psi, \tag{17}$$

where $\boldsymbol{\lambda} \equiv \begin{bmatrix} \lambda_d & \lambda_\alpha & \lambda_\beta \end{bmatrix}$ is a vector of adjoint variables. The adjoint dynamics are given by [4] $\dot{\boldsymbol{\lambda}} = -\frac{\partial \mathscr{H}}{\partial \mathbf{z}}$, which simplifies to

$$\dot{\lambda}_d = (\lambda_\beta - \lambda_\alpha)\frac{1}{d^2}\sin\psi, \quad \dot{\lambda}_\alpha = 0, \quad \dot{\lambda}_\beta = 0, \tag{18}$$

thus λ_α and λ_β are constant. At $t = t_l$, the adjoint vector is [4]

$$\boldsymbol{\lambda}^\top(t_l) = \frac{\partial \Phi_l}{\partial \mathbf{z}_l} + \nu \frac{\partial \phi_l}{\partial \mathbf{z}_l}, \tag{19}$$

where ν is another adjoint variable. The value of the Hamiltonian at the time of lock is [4]

$$\mathscr{H}(t_l) = -\frac{\partial \Phi_l}{\partial t_l} - \nu \frac{\partial \phi_l}{\partial t_l} \tag{20}$$

When the functions Φ_l and ϕ_l do not depend on terminal time $\mathscr{H}_l = 0$, and since the system kinematics are time-autonomous, $\mathscr{H}(t) = 0$ for all $t \in [t_0, t_l]$.

A and D's equilibrium control strategies minimize and maximize the Hamiltonian, respectively:

$$\psi^* = \arg\min_\psi \mathscr{H}, \qquad \omega^* = \arg\max_\omega \mathscr{H},$$

which becomes

$$\cos\psi^* = \frac{-\lambda_d}{\sqrt{\lambda_d^2 + \frac{(\lambda_\alpha - \lambda_\beta)^2}{d^2}}}, \quad \sin\psi^* = \frac{\lambda_\alpha - \lambda_\beta}{d\sqrt{\lambda_d^2 + \frac{(\lambda_\alpha - \lambda_\beta)^2}{d^2}}} \tag{21}$$

$$\omega^* = \rho \cdot \operatorname{sign}\lambda_\alpha. \tag{22}$$

The value of the adjoint variable ν is generally determined by evaluating the Hamiltonian, (17), at lock time and substituting in (19)–(22).

In each case, $\operatorname{sign}\lambda_\alpha$ is not fully determined by the above first order necessary conditions for equilibrium. Consequently, there arises a Dispersal Surface (c.f. [1], [3]) which partitions the region of admissible initial conditions into a region where D either moves CCW or CW. The region of admissible initial conditions may be constructed via backwards integration of the equilibrium kinematics ((21) and (22) substituted into (2)) from the terminal manifold (or some particular limiting manifold, depending on the case). To simplify the process of constructing these regions, we state the following

Lemma 1. *The Attacker's equilibrium trajectory for the pre-lock segment of the LEER game, $t \in [0, t_l]$, is a straight line in the Cartesian frame.*

Proof. The result follows from the simple motion (single integrator dynamics) of A along with the fact that the cost functional in (16) is of Mayer-type (i.e., only a function of the terminal conditions). The details are omitted; a similar style proof is contained in [11]. □

Thus, A's initial position can be obtained using its terminal Cartesian heading $\tilde{\psi}_l^* = \psi_l^* + \beta_l$ for a particular t_l.

IV. The Post-Lock Engage or Retreat Game

This section of the paper addresses the definition and solution of the post-lock segment of the LEER game, which is needed to solve the pre-lock segment (as dictated by (14)). The Engage or Retreat Game with an Attacker moving with simple motion and a Defender whose control appears directly in the Attacker's integral cost was formulated and solved in [5]. This corresponds to D having a lock on A over the entire game, i.e., $L = 1$ for all $t \in [t_0, t_f]$.

When A chooses engagement, D incurs its maximum cost and has "lost" in some sense. So, generally, D seeks to make engagement as costly for A as possible to make retreat a more attractive option for A. Thus the agents play a zero-sum differential game with J_A as a cost functional; the associated Value function of the differential Game of Engagement (GoE) is defined as

$$V_E\left(\mathbf{z}_0\right) = \min_{\mathbf{u}_A(\cdot)} \max_{\mathbf{u}_D(\cdot)} J_A\left(\mathbf{z}_0; \mathbf{u}_A(\cdot), \mathbf{u}_D(\cdot)\right), \tag{23}$$

with the constraint $\phi\left(\mathbf{z}_f\right) = d_f - 1 = 0$. The solution of the game is comprised of the following expression for the Value function [5]

$$V_E\left(\mathbf{z}\right) = \left(\bar{w} + c\right)\left(d - 1\right) \tag{24}$$

along with the state-feedback equilibrium strategies [5]

$$\psi^*\left(\mathbf{z}\right) = \pi, \qquad w^*\left(\mathbf{z}\right) = \bar{w}, \tag{25}$$

which corresponds to A aiming directly at the D and D applying its maximum defense.

For retreat, D wants to encourage A to continue retreating and thus seeks to minimize A's cost. The Value function for the associated optimal control problem is defined as

$$V_R\left(\mathbf{z}_0\right) = \min_{\mathbf{u}_A(\cdot), \mathbf{u}_D(\cdot)} J_A\left(\mathbf{z}_0; \mathbf{u}_A(\cdot), \mathbf{u}_D(\cdot)\right), \tag{26}$$

with the constraint $\phi\left(\mathbf{z}_f\right) = d_f \sin\beta_f - y_R = 0$ which corresponds to $y_f = y_R$. An additional constraint is necessary in order to ensure the validity of the Optimal Constrained Retreat (OCR) trajectory:

$$V_E\left(\mathbf{z}(t)\right) - V_R\left(\mathbf{z}(t)\right) \geq 0 \quad \forall t \in [t_0, t_f], \tag{27}$$

which essentially requires that engagement must be as costly or more costly than retreating along the entire retreat trajectory. The manifold $V_E(\mathbf{z}) = V_R(\mathbf{z})$ partitions Ω into a region where engagement is optimal and a region where retreat is optimal. The latter may be further partitioned depending on whether or not A must maneuver around the engagement region in order to retreat (which occurs when (27) is activated). Let the partitioning of Ω be defined as

$$\mathscr{R}_E = \{\mathbf{z} \mid V_E(\mathbf{z}) < V_R(\mathbf{z})\} \tag{28}$$

$$\mathscr{R}_{R_1} = \{\mathbf{z} \notin \mathscr{R}_E \mid |x| > x_2 \text{ or } y < y_2\} \tag{29}$$

$$\mathscr{R}_{R_2} = \Omega \setminus \left(\mathscr{R}_E \cup \mathscr{R}_{R_1}\right) \tag{30}$$

where

$$x_2 = \frac{\bar{w} + c + c_A - c y_R}{\sqrt{\left(\bar{w} + c\right)^2 - c^2}}, \; y_2 = \frac{c\left(\bar{w} + c + c_A - c y_R\right)}{\left(\bar{w} + c\right)^2 - c^2}. \tag{31}$$

Figure 2 shows the partitioning of the state space for the parameter settings used throughout the remainder of the paper. In $\mathscr{R}_{R_1}$, the solution is comprised of the Value

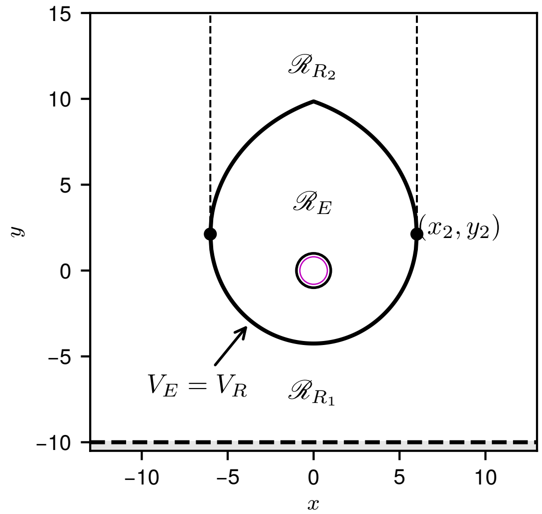

Fig. 2. Engage or retreat regions. $c = 0.5$, $c_A = 2$, $\rho = 0.8$, $\bar{w} = 1$, and $y_R = -10$.

function expression [6]

$$V_{R_1} \equiv V_R\left(\mathbf{z}\right) = c\left(d \sin\beta - y_R\right) + c_A, \; \mathbf{z} \in \mathscr{R}_{R_1} \tag{32}$$

along with the state-feedback equilibrium strategies [6]

$$\psi^*(\mathbf{z}) = \frac{3\pi}{2} - \beta, \qquad w^*(\mathbf{z}) = 0, \tag{33}$$

which corresponds to A running straight down and D holding fire. In $\mathscr{R}_{R_2}$, the solution is comprised of the Value function expression

$$V_{R_2} \equiv V_R\left(\mathbf{z}\right) = c t_f^* + c_A, \; \mathbf{z} \in \mathscr{R}_{R_2} \tag{34}$$

where t_f^* is determined numerically; the details are contained in [6].

Finally, the overall Value function for the post-lock segment of the LEER game is defined as

$$V_{EoR}\left(\mathbf{z}\right) \equiv \begin{cases} V_E\left(\mathbf{z}\right) & \mathbf{z} \in \mathscr{R}_E \\ V_{R_1}\left(\mathbf{z}\right) & \mathbf{z} \in \mathscr{R}_{R_1} \\ V_{R_2}\left(\mathbf{z}\right) & \mathbf{z} \in \mathscr{R}_{R_2} \end{cases} \tag{35}$$

V. Locked Engagement

In this section we analyze the case in which the LEER game terminates in engagement. Again, the post-lock segment corresponds to the classical Engage or Retreat Game discussed in the previous section. Thus in order for engagement to be optimal Attacker behavior, it is necessary for the state of the system at the time of lock-on to be in the engage region. We define the region for which lock-on occurs and engagement is optimal in the LEER game as

$$LE \equiv \left\{\mathbf{z} \mid \mathbf{z}_l^* \in \mathscr{R}_E\right\}. \tag{36}$$

Note the region LE is conditioned on $\mathbf{z}_l^*$ – the (equilibrium) state of the system when D aligns its turret with

A, which is the terminal state for the pre-lock segment. Therefore, we will utilize the first order necessary conditions for equilibrium, derived below, to construct the region LE via backwards integration.

By construction $\mathbf{z}_l \in \mathscr{R}_E$ and thus the terminal cost function is obtained by substituting (24) into (14)

$$\Phi_l(\mathbf{z}_l, t_l) = V_E(\mathbf{z}_l) = (\bar{w} + c)(d_l - 1), \tag{37}$$

with the terminal boundary condition given in (15). The terminal adjoint values are obtained by substituting (37) and (15) into (19)

$$\lambda_{d_l} = \bar{w} + c, \quad \lambda_\alpha = \nu, \quad \lambda_\beta = 0 \tag{38}$$

Evaluating (27) at $t = t_l$ and substituting in (38) yields the following pre-lock terminal equilibrium controls

$$\cos\psi_l^* = \frac{-(\bar{w}+c)}{\sqrt{(\bar{w}+c)^2 + \frac{\nu^2}{d_l^2}}}, \ \sin\psi_l^* = \frac{\nu}{d_l\sqrt{(\bar{w}+c)^2 + \frac{\nu^2}{d_l^2}}} \tag{39}$$

$$\omega^* = \rho \operatorname{sign} \nu. \tag{40}$$

Note ω^* is constant over $t \in [0, t_l]$ because ν is constant. Evaluating (17) at $t = t_l$, substituting in the terminal adjoints (38) and terminal equilibrium controls (39), (40), and setting equal to 0 allows ν to be obtained:

$$\nu = \frac{\pm(\bar{w}+c)\, d_l}{\sqrt{\rho^2 d_l^2 - 1}} \tag{41}$$

Lemma 2. *The terminal equilibrium controls for Locked Engagement are given by*

$$\cos\psi_l^* = \frac{-\sqrt{\rho^2 d_l^2 - 1}}{\rho d_l}, \ \sin\psi_l^* = \frac{-\operatorname{sign}(\sin\alpha_0)}{\rho d_l} \tag{42}$$

$$\omega^* = -\rho \operatorname{sign}(\sin\alpha_0), \tag{43}$$

Proof. We begin by proving that $\operatorname{sign}\nu = -\operatorname{sign}(\sin\alpha_0)$. D wishes to maximize (37) and thus seeks to drive $\alpha \to 0$ with maximum d. Meanwhile, A minimizes (37) and wishes to terminate as close to D as possible. From (21) $\cos\psi^* < 0$ and thus d is monotonically decreasing over $t \in [0, t_l]$. Therefore, D must drive $\alpha \to 0$ as quickly as possible, which corresponds to turning at its maximum angular rate towards A: $\omega^* = -\rho \operatorname{sign}(\sin\alpha_0)$. Equating this expression to (40) yields $\operatorname{sign}\nu = -\operatorname{sign}(\sin\alpha_0)$. Substituting this along with (41) into (39) and (40) yields (42) and (43). □

Corollary 1. *In the region LE there exists a Dispersal Surface wherein the Defender and Attacker may either turn CCW or CW and achieve the same Value in the LEER game, and is defined by*

$$\mathcal{D}_{LE} \equiv \{\mathbf{z} \mid \mathbf{z} \in LE, \ \cos\alpha = -1\}. \tag{44}$$

Proof. The Dispersal Surface $\mathcal{D}_{LE}$ arises due to the symmetry in the problem geometry and the equilibrium controls. When $\cos\alpha = -1$ we have $\sin\alpha = 0$ and thus the direction of the equilibrium controls, (42) and (43) is undefined. The pre-lock segment terminates at a particular d_l with $\alpha_l = 0$. Thus, the initial state wherein $\cos\alpha = -1$ can be reached via backwards integration with CCW or CW motion. These two trajectories have the same Value since V_E is only a function of d. This proof is similar to one used in [12]; the interested reader is referred therein for further detail. □

VI. Locked Retreat

There are two regions, $\mathscr{R}_{R_1}$ and $\mathscr{R}_{R_2}$, in which A would choose to retreat. Trajectories beginning in $\mathscr{R}_{R_1}$ are unconstrained, while those beginning in $\mathscr{R}_{R_2}$ are constrained. The solutions will be discussed in Sections VI-A and VI-B, respectively.

A. LR ending in $\mathscr{R}_{R_1}$

Similar to the previous section, we define a region where retreat is optimal in the LEER game, lock occurs, and the retreat trajectory is unconstrained (i.e., (27) remains inactive):

$$LR_1 \equiv \{\mathbf{z} \mid \mathbf{z}_l^* \in \mathscr{R}_{R_1}\}. \tag{45}$$

Substituting (32) into (14) yields the terminal cost function

$$\Phi_l(\mathbf{z}_l, t_l) = V_{R_1}(\mathbf{z}_l) = c(d_l \sin\beta_l - y_R) + c_A. \tag{46}$$

Lemma 3. *The terminal equilibrium controls for Locked Retreat ending in $\mathscr{R}_{R_1}$ are given by*

$$\cos\psi_l^* = \frac{-c\sin\beta_l}{\chi}, \ \sin\psi_l^* = \frac{-c\cos\beta_l + \frac{\nu}{d_l}}{\chi} \tag{47}$$

$$\omega^* = \rho \operatorname{sign}\nu, \tag{48}$$

where

$$\nu = -\frac{c d_l \cos\beta_l}{\rho^2 d_l^2 - 1} \pm c d_l \sqrt{\frac{c d_l \cos^2\beta_l}{(\rho^2 d_l^2 - 1)^2} + \frac{1}{\rho^2 d_l^2 - 1}}, \tag{49}$$

$$\chi = \sqrt{c^2 \sin^2\beta_l + \left(c\cos\beta_l - \frac{\nu}{d_l}\right)^2}. \tag{50}$$

Proof. The proof follows along the process described in Section III and demonstrated in Section V; the details are omitted. □

Corollary 2. *In the region LR_1 there exists a Dispersal Surface, $\mathcal{D}_{LR_1}$, wherein the Defender and Attacker may either turn CCW or CW and achieve the same Value in the LEER game.*

Proof. The logic is similar to Corollary 1, however, the polar symmetry from the LE case is lost in the LR case since A is ultimately heading for the retreat zone. Hence we have no analytic expression for the surface $\mathcal{D}_{LR_1}$, and as a result, the surface is computed numerically. □

The Dispersal Surface $\mathcal{D}_{LR_1}$ partitions LR_1 into two regions wherein D has either CCW motion or CW motion (which corresponds to the two sign possibilities for ν in (49)). Computation of $\mathcal{D}_{LR_1}$ is thus useful for determining the sign of ν for particular terminal conditions. This computation is accomplished via coupled backwards shooting process adapted from [13] wherein we obtain pairs of trajectories which have equal Value and integrate back to the same initial condition.

Special care must be taken when backwards integrating as the state of the system may enter a region where Locked Engagement becomes optimal. Figure 3 shows an example trajectory for LR and LE, $\mathbf{z}_{LR}$ and $\mathbf{z}_{LE}$, respectively, which have the same Value. The manifold for which the Value of LE and LR are identical is akin to the $V_E = V_R$ manifold shown in Fig. 2 except that it is a 2D manifold in the 3D state space.

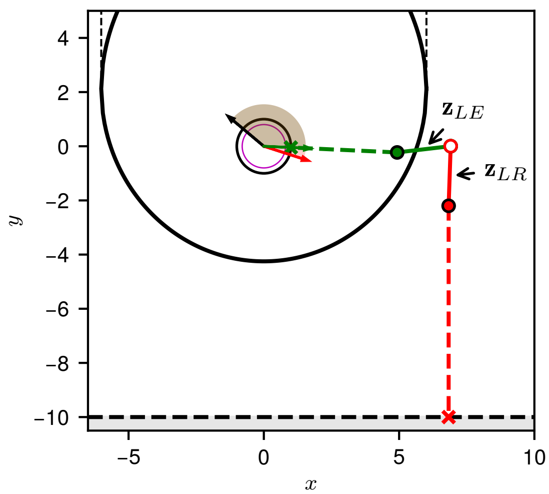

Fig. 3. Example trajectories for LE (green) and LR (red) with the same Value. The open circle is the initial A position, closed circles are the A position when lock-on occurs. The post-lock trajectories are shown in dashed lines and an $\times$ marks A's position at t_f. Initially, D's turret is aimed along the black vector.

B. LR ending in $\mathscr{R}_{R_2}$

As in previous sections, we define a region where retreat is optimal in the LEER game, lock occurs, and the retreat trajectory is constrained (i.e., (27) is activated):

$$LR_2 \equiv \{\mathbf{z} \mid \mathbf{z}_l^* \in \mathscr{R}_{R_2}\}. \tag{51}$$

For convenience, the analysis is done in the Cartesian frame for this case. Substituting (34) into (14) yields the terminal cost function

$$\Phi_l(\tilde{\mathbf{z}}_l, t_l) = V_{R_2}(\tilde{\mathbf{z}}_l) = ct_f^* + c_A. \tag{52}$$

In the Cartesian frame, the terminal manifold (15) becomes

$$\phi(\tilde{\mathbf{z}}_l, t_l) = \gamma_1 - \arctan\left(\frac{y_l}{x_l}\right) = 0. \tag{53}$$

The solution for this case is complicated by the lack of an analytical expression for t_f^*, which is the optimal time for A to reach the retreat zone while satisfying (27). From [6], the optimal constrained retreat trajectory for A is comprised of a straight segment which is tangent to the $V_R = V_E$ manifold, a curved segment which rides along the manifold, and a straight segment departing from (x_2, y_2) terminating at (x_2, y_R). Let $\tilde{\psi}_R^*$ be the optimal retreat heading for A at $t = t_l$ which is tangent to the $V_R = V_E$ manifold. Then, specializing (19) the pre-lock terminal adjoint values are

$$\boldsymbol{\lambda}(t_l) = \frac{\partial V_{R_2}}{\partial \tilde{\mathbf{z}}} + \nu \frac{\partial \phi_l}{\partial \tilde{\mathbf{z}}} \tag{54}$$

$$\implies \lambda_{x_l} = -c\cos\tilde{\psi}_R^* + \nu \frac{y_l}{x_l^2 + y_l^2} \tag{55}$$

$$\lambda_{y_l} = -c\sin\tilde{\psi}_R^* - \nu \frac{x_l}{x_l^2 + y_l^2} \tag{56}$$

$$\lambda_{\gamma_l} = \nu. \tag{57}$$

Note that $\frac{\partial V_{R_2}}{\partial \tilde{\mathbf{z}}}$ is derived in the Appendix.

Lemma 4. *The terminal equilibrium controls for Locked Retreat ending in $\mathscr{R}_{R_2}$ are given by*

$$\cos\tilde{\psi}_l^* = \frac{-\lambda_{x_l}}{\sqrt{\lambda_{x_l}^2 + \lambda_{y_l}^2}}, \quad \sin\tilde{\psi}_l^* = \frac{-\lambda_{y_l}}{\sqrt{\lambda_{x_l}^2 + \lambda_{y_l}^2}} \tag{58}$$

$$\omega^* = \rho \operatorname{sign} \nu \tag{59}$$

where $\lambda_{x_l}, \lambda_{y_l}, \lambda_{\gamma_l}$ are given in (55)–(57), $\tilde{\psi}_R^*$ is computed numerically, and

$$\nu = \frac{-b \pm \sqrt{b^2 - 4ac^2}}{2a} \tag{60}$$

where

$$a = \left(\frac{1}{x_l^2 + y_l^2} - \rho^2\right) \tag{61}$$

$$b = 2c\left(\frac{y_l\cos\tilde{\psi}_R^* - x_l\sin\tilde{\psi}_R^*}{x_l^2 + y_l^2}\right). \tag{62}$$

Proof. The proof follows along the process described in Section III but in the Cartesian frame; the details are omitted for space. □

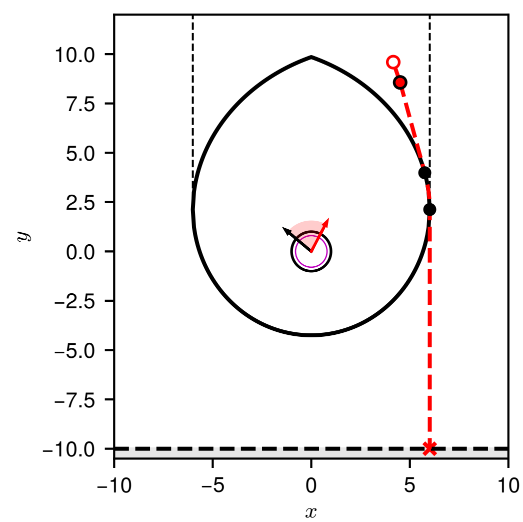

Fig. 4. Example trajectory for LR ending in $\mathscr{R}_{R_2}$. The initial turret look angle is shown by the black arrow. A starts at the open circle, gets locked-onto at the closed red circle, enters the constrained arc at the upper black circle, leaves the $V_E = V_R$ manifold at the lower black circle, and reaches the retreat zone at the $\times$.

VII. Full Solution

In this section we utilize the results of the previous sections to develop the solution over the whole state space. In particular, the state space is partitioned into 5 distinct regions in which a particular A behavior and termination is optimal – LE, LR ending in $\mathscr{R}_{R_1}$ or $\mathscr{R}_{R_2}$, UE, and UR. Along the boundaries of these regions, A may have two or more choices which yield the same Value, J_A^*, in the LEER game. Because the state space Ω is in 3D, the regions are 3D as well, and the boundaries are 2D manifolds in 3D space. In order to visualize the partitioning in a more meaningful way, we fix the initial look angle of D's turret. Then the regions correspond to initial A positions (in 2D) and the boundaries become 1D. For a particular initial turret look angle, γ_0, the 1D boundaries are obtained by identifying the appropriate terminal manifold(s) and extending A's position backwards in time using the associated terminal equilibrium heading until $\gamma = \gamma_0$.

For Locked Engagement there are two critical configurations to consider: 1) lock-on occurs exactly when A reaches D ($d_l = 1$) and 2) lock-on occurs exactly when $V_E = V_R$. The first is critical in the sense that, for LE, we require that lock-on actually occur; the limiting case is when it occurs just before the overall LEER game terminates. The boundary obtained via backwards integration from $d_l = 1$ divides the LE and UE regions – that is, if A is initialized *inside* this boundary, A can reach D and avoid being locked-on. The terminal manifold $V_E = V_R$ is critical because for engagement to be optimal, A must be in $\mathscr{R}_E$ by the time lock-on occurs; the limiting case is A just barely reaches $\mathscr{R}_E$ when $\cos\alpha \to 1$. Backwards integration from this terminal manifold divides the LE and LR regions.

For Locked Retreat, we further distinguish LR ending in $\mathscr{R}_{R_1}$ or $\mathscr{R}_{R_2}$. LR ending in $\mathscr{R}_{R_1}$ has one critical terminal manifold: lock-on occurs exactly when A reaches the retreat zone ($y_l = y_R$). The associated boundary divides LR ending in $\mathscr{R}_{R_1}$ from UR – that is, if A is initialized *below* this boundary, A can reach y_R and avoid being locked-on. LR ending in $\mathscr{R}_{R_2}$ has one critical terminal manifold in which lock-on occurs exactly when A reaches the boundary between $\mathscr{R}_{R_1}$ and $\mathscr{R}_{R_2}$ ($|x_l| = x_2$, $y_l > y_2$). The associated boundary merely distinguishes between LR ending in $\mathscr{R}_{R_2}$ versus $\mathscr{R}_{R_1}$.

The terminal manifolds identified above do not all have the same cost. Depending on the problem parameters (ρ, c, $\bar{w}$), points on the boundary between LE and LR must be obtained by integrating back from equi-Valued terminal manifolds (instead of the critical terminal manifolds identified above). Figures 5 and 6 each show an example partitioning for a particular γ_0.

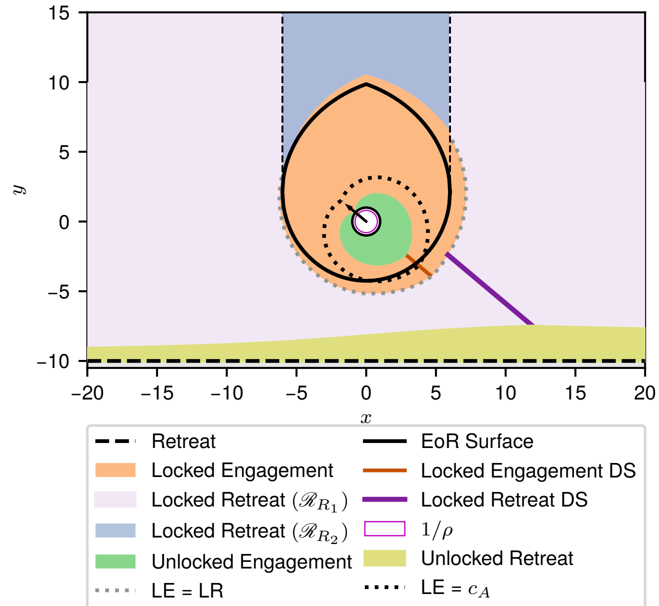

Fig. 5. A partitioning of the state space for a particular initial turret look angle, $\gamma_0 = \frac{7}{9}\pi$.

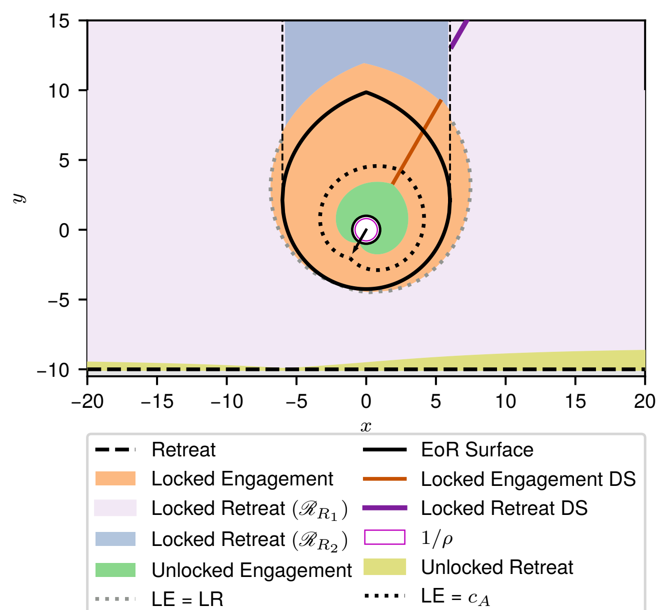

Fig. 6. A partitioning of the state space for a particular initial turret look angle, $\gamma_0 = \frac{12}{9}\pi$.

VIII. Conclusion

In this paper we extended the canonical engage or retreat game by considering the Defender to have a turn-constrained turret. The focus was on the cases in which the Defender could aim its turret onto the Attacker's position prior to the Attacker reaching either the engagement or retreat terminal surfaces. For Locked Engagement and Locked Retreat, the equilibrium control strategies for the Attacker and Defender were derived as a function of the terminal state. Using backwards integration, these strategies were used to construct a partitioning of the state space whose regions correspond to the various terminal cases.

Extensions to this work include a fuller treatment of the Unlocked Engagement and Unlocked Retreat cases – in particular, a means by which to generate valid UE and UR trajectories (which can be guaranteed avoid turret lock-on). Additionally, the characterization and construction of the Dispersal Surface existing in the constrained Locked Retreat region is necessary for a full solution. Lastly, this paper presents results for a particular set of parameters; it would be worthwhile to characterize the solution across a range of parameters.

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APPENDIX

This appendix contains a derivation of the constrained retreat Value partial derivatives $\frac{\partial V_{R_2}}{\partial \tilde{\mathbf{z}}}$. Let $\boldsymbol{\sigma} \equiv \frac{\partial V_{R_2}}{\partial \tilde{\mathbf{z}}} = \begin{bmatrix}\sigma_x & \sigma_y & \sigma_\gamma\end{bmatrix}^\top$. We are concerned with the post-lock segment wherein lock-on has occurred, and thus γ no longer has any bearing on the optimality since D has sufficient control authority to maintain the lock-on (i.e., keep $\cos\alpha = 1$). Therefore, $\sigma_\gamma = 0$. Thus the Hamiltonian for the retreat case in the Cartesian frame is

$$\mathscr{H}_R = \sigma_x \cos\tilde{\psi}_R + \sigma_y \sin\tilde{\psi}_R \tag{63}$$

Here, the terminal cost function is simply $\Phi_R(\tilde{\mathbf{z}}_f, t_f) = ct_f + c_A$, and the terminal manifold is $\phi_R(\tilde{\mathbf{z}}_f, t_f) = y_f - y_R = 0$ [6]. The value of $\mathscr{H}_R$ at the post-lock termination is [4]

$$\mathscr{H}_R(t_f) = -\frac{\partial \Phi_R}{\partial t_f} - \mu \frac{\partial \phi_R}{\partial t_f} = -c. \tag{64}$$

The Attacker must minimize $\mathscr{H}_R$, and thus

$$\cos\tilde{\psi}_R^* = \frac{-\sigma_x}{\sqrt{\sigma_x^2 + \sigma_y^2}}, \quad \sin\tilde{\psi}_R^* = \frac{-\sigma_y}{\sqrt{\sigma_x^2 + \sigma_y^2}}. \tag{65}$$

Evaluating (65) at $t = t_f$ and substituting into (63) gives

$$\mathscr{H}_R(t_f) = -\sqrt{\sigma_x^2 + \sigma_y^2} = -c. \tag{66}$$

Substituting (66) into (65) yields the desired expressions:

$$\sigma_x = -c\cos\tilde{\psi}_R^*, \quad \sigma_y = -c\sin\tilde{\psi}_R^*. \tag{67}$$