The Turret-Runner-Penetrator Differential Game

Alexander Von Moll[1,2], Daigo Shishika[3], Zachariah Fuchs[2], and Michael Dorothy[4]

***Abstract*— A scenario is considered in which two cooperative Attackers aim to infiltrate a circular target guarded by a Turret. The engagement plays out in the two dimensional plane; the holonomic Attackers have the same speed and move with simple motion and the Turret is stationary, located at the target circle's center, and has a bounded turn rate. When the Turret's look angle is aligned with an Attacker, that Attacker is terminated. In this paper, we focus on a region of the state space wherein only one of the Attackers is able to reach the target circle – and even then, only with the help of its partner Attacker. The Runner distracts the Turret and ends up being terminated in order that the Penetrator can be guaranteed to hit the target circle. We formulate the Turret-Runner-Penetrator scenario as a differential game over the Value of the subsequent game of $\min\max$ terminal angle which takes place between the Turret and Penetrator once the Runner has been terminated. The solution to the Game of Degree, including equilibrium Turret, Runner, and Penetrator strategies, as well as the Value function are given. In addition, the Game of Kind solution, which is the manifold of states in which the Penetrator will be terminated exactly on the target circle, is constructed numerically.**

I. Introduction

Cooperation is essential for success in conflicts between teams of agents. Certain outcomes are only possible through cooperation; victory could even be contingent on the sacrifice of a particular agent. In this paper, we consider a cooperative team of Attackers who seek to collide with a static target that is guarded by a Turret. The Turret is equipped with a directional weapon which can be aimed (turned) with bounded rate. When the Turret's look angle is aligned with the position of an Attacker, that Attacker is considered to be destroyed (or terminated). Our focus is on providing a rigorous solution for the case where there are two Attackers. This is a step towards analyzing the defense of a static location or asset against "swarms" of Attackers with a directional defensive weapon. In particular, we aim to consider the many Attacker case wherein the Turret must destroy all (or as many as is possible) Attackers in succession; the Attackers, meanwhile, coordinate their attack to maximize successful hits. With the prospective proliferation of lower-cost unmanned vehicles and guided munitions solving this problem is of interest, both from the attackers' and defender's perspective. See, for example, the following excerpt from the Air Force 2030 Science & Technology Strategy document [1] (emphasis added):

> Swarms of low-cost, autonomous air and space systems can ... **absorb losses** that manned systems cannot... Low-end systems can restore the agility to attack adversary weaknesses in unexpected ways by **exploiting numbers** and complexity.

Various turret and turret-like defense scenarios have been explored in recent literature. The works vary in aspects such as number of agents, cost functional (particularly integral versus terminal), and termination conditions. However, the agents' kinematics are essentially the same with the slight exception of the Turret/Defender. In some cases, the Defender is modeled as an agent with bounded speed who is constrained to move along the perimeter of the target circle, and in others, the Defender is stationary and turn-constrained Turret; these two models are equivalent. In [2], the authors formulated and solved the Turret Defense Differential Game (along with all of its singularities) wherein the cost functional included a state-dependent integral cost. There, a single mobile Attacker sought to balance time-to-target with avoiding the line of sight of the Turret; the resulting Attacker trajectories are generally curved in the Cartesian frame. Reference [3] analyzed a perimeter patrol scenario wherein termination occurs either when the Attacker reaches the target or when the Defender and Attacker are coincident. The solution characteristics of the one-Attacker, one-Defender and one-Attacker, two-Defender scenarios were then extended to a many-Attacker, many-Defender variant wherein the teams maximize (minimize, respectively) the number of hits on the target. An extension considered a heterogeneous Defender team comprised of uncontrolled and controlled patrollers [4]. In [5], [6], the authors solved a similar scenario but with turret-style termination conditions (i.e. line of sight termination) for the one-Attacker, one-Turret and one-Attacker, two-Turret cases. This paper is an extension thereof in which we consider aspects of the two-Attacker, one-Turret case.

We consider a particular sub-case wherein one of the Attackers must sacrifice itself in order for the other Attacker to reach the target unhindered. Because the Attackers essentially have different roles, this problem is also related to other "three-body" problems in the literature, such as the Target Attacker Defender Differential Game [7]. There, the

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[1]Control Science Center of Excellence, Air Force Research Laboratory, 2210 8th St. WPAFB, OH 45433 alexander.von_moll@us.af.mil

[2]Department of Electrical Engineering and Computing Systems, University of Cincinnati, Cincinnati, OH 45221

[3]Department of Mechanical Engineering, George Mason University, Fairfax, VA 22030

[4]Combat Capabilities Development Command, Army Research Laboratory, Adelphi, MD, 20783

Defender/Target team seek to cooperatively maneuver in such a way for the Defender to intercept the Attacker as far from the Target as possible. Another example is the single-pursuer, two-evader cooperative defense scenario presented in [8] wherein one of the Evaders performs a flanking maneuver on the Pursuer to drive up the Pursuer's cost. The work in this paper is also related to the problem of capture of two evaders in succession [9] since the Turret is free to aim at another Attacker once one is terminated.

The two-Attacker, one-Turret problem is formulated and solved (for a particular region of the state space) using the framework of differential game theory (c.f. [10]). In particular, we address the case in which neither Attacker can guarantee to reach the target individually, but, through their cooperation, the Attackers can guarantee that one can. We thus pose and solve the Turret-Runner-Penetrator Differential Game (TRPDG), providing both the Value function and equilibrium strategies. Section II provides a formulation for the overall two-Attacker, one-Turret problem and breaks the general problem up based on how many Attackers can be guaranteed to reach the target. Section III specifies the TRPDG which takes place in the state space region where exactly one Attacker can be guaranteed to reach the target. Its subsections III-A, III-B, and III-C contain the derivation of the Turret and Runner strategies, the Penetrator strategy, and the full solution, respectively. Section IV provides some conclusions and identifies specific problems to address in future work.

II. Problem Formulation

In this formulation, the speed of the two Attackers are equal. Let $\nu < 1$ be the ratio of the Attackers' speed and Turret's maximum turn rate, the latter of which is normalized to 1. Let $\hat{\mathbf{z}} = (x_1, y_1, x_2, y_2, \beta)$ be the state of the system wherein the two Attackers' positions are represented by their 2-D Cartesian coordinates and the Turret's look angle is β w.r.t. the positive x-axis. The target circle has a perimeter of 1; thus the Turret has an angular velocity advantage. The kinematics are thus

$$\hat{f}(\hat{\mathbf{z}}) = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \nu \cos \hat{\psi}_1 \\ \nu \sin \hat{\psi}_1 \\ \nu \cos \hat{\psi}_2 \\ \nu \sin \hat{\psi}_2 \\ u_T \end{bmatrix}, \tag{1}$$

where $\hat{\psi}_1, \hat{\psi}_2 \in [0, 2\pi]$ are the Attackers' headings measured w.r.t. the positive x-axis, and $u_T \in [-1, 1]$ is the Turret's angular velocity input (with positive u_T corresponding to counterclockwise motion). Alternatively, the Attackers' positions may be expressed in a polar coordinate system centered on the target circle's center. Define $\mathbf{z} = (R_1, \theta_1, R_2, \theta_2, \beta)$ where $\theta_1, \theta_2 \in [-\pi, \pi]$ are measured relative to the Turret's look angle. Also let $A_1 \equiv (R_1, \theta_1)$ and $A_2 \equiv (R_2, \theta_2)$; the Turret is also denoted T. The associated kinematics are

$$f(\mathbf{z}) = \begin{bmatrix} \dot{R}_1 \\ \dot{\theta}_1 \\ \dot{R}_2 \\ \dot{\theta}_2 \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -\nu \cos \psi_1 \\ \frac{\nu}{R_1} \sin \psi_1 - u_T \\ -\nu \cos \psi_2 \\ \frac{\nu}{R_2} \sin \psi_2 - u_T \\ u_T \end{bmatrix} \tag{2}$$

where $\psi_1, \psi_2 \in [-\pi, \pi]$ are measured clockwise w.r.t. the line from the respective Attacker to the target circle center. Figure 1 depicts the scenario, showing both coordinate systems specified above.

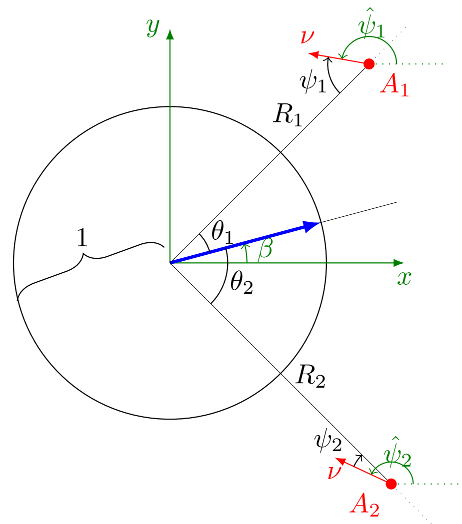


Fig. 1. Two Attacker Scenario – the green color indicates the Cartesian coordinate system; black represents the polar coordinate system. The Attacker position angles, θ_1 and θ_2, are measured w.r.t. T's look angle and are positive in the CCW direction (thus $\theta_2 < 0$, as shown).

An Attacker A_i is considered to be terminated (and removed from the remainder of the playout, if any) if at any time $\theta_i = 0$. Conversely, A_i is said to 'win' if it can maneuver all the way to the target circle ($R_i = 1$) while avoiding the Turret's line-of-sight.

There are three cases: (i) both Attackers win, (ii) one Attacker is terminated and one wins, or (iii) both Attackers are terminated. Cases (i) and (iii) are discussed briefly in the Appendix. The remainder of the paper focuses on the state space region wherein $A_1, A_2 \notin \mathscr{R}_A$ at initial time, where $\mathscr{R}_A$ is the one-Attacker, one-Turret Attacker's win region, defined in (31) in the Appendix. In this region, neither Attacker can guarantee a win against the Turret; we will construct a subset of this region in which, through their cooperation and superiority in numbers, one of the Attackers can win. Let the region of interest for this state space be defined as

$$\Omega := \{\mathbf{z} \mid A_1, A_2 \notin \mathscr{R}_A, \; R_1, R_2 > 1\}. \tag{3}$$

III. The Turret-Runner-Penetrator Differential Game

We now proceed with the analysis in the polar coordinate system, utilizing (2), with $\mathbf{z} \in \Omega$. A major assumption is made at this point, which is that the fate of each Attacker is set *a priori*. Let A_1 be the first Attacker to be terminated by T, regardless of the position of A_2. A complete solution, which involves the agents determining which Attacker T will pursue and terminate first, necessarily depends on the solution of this simpler problem. We begin by assuming that A_2 *can* reach the target circle ($R_2 = 1$) and that A_2 prefers to maximize its angular separation from T at final time. That is, in the second phase of the engagement that begins when A_1 is terminated, A_2 plays the *Game of Angle*, as specified in [5]. As such, we refer to A_1 as the Runner, and A_2 as the Penetrator. The construction and solution of a *Game of Degree* wherein A_2 is also terminated is left for future work. As this setup implicitly assumes that A_2 has not reached the target circle by the time A_1 is terminated, the construction and solution of a *Game of Degree* for such a case is left for future work. We model the *first* phase of the engagement as a zero-sum differential game over the cost functional

$$J = \Phi\left(\mathbf{z}(t_c), t_c\right) = V_{A_2}\left(R_2(t_c), \theta_2(t_c)\right) = \min_{u_T(\cdot)} \max_{\psi_2(\cdot)} \theta_2(t_f) \tag{4}$$

where V_{A_2} is the Value of the *Game of Angle* played between A_2 and T starting from $t = t_c$, and t_c is the terminal time of the game, which occurs when A_1 is terminated. Thus we define the terminal manifold as

$$\phi\left(\mathbf{z}(t_c), t_c\right) = \theta_1(t_c) = 0. \tag{5}$$

The Attackers cooperatively seek to maximize J, while the Turret wants to minimize J. Thus the Value function for the Turret-Runner-Penetrator Differential Game (TRPDG) is defined as

$$V\left(\mathbf{z}; u_T(\cdot), \psi_1(\cdot), \psi_2(\cdot)\right) = \min_{u_T(\cdot)} \max_{\psi_1(\cdot), \psi_2(\cdot)} J \tag{6}$$

The Value function of the *Game of Angle* is given in [5] as

$$V_{A_2}\left(R_2, \theta_2\right) = \theta_2 - \theta_{GoK}\left(R_2\right), \tag{7}$$

where θ_{GoK} is the one-Attacker, one-Turret *Game of Kind* surface defined in (32) in the Appendix. Figure 2 depicts the overall scenario broken up into two distinct phases: Phase 1, which terminates at $t = t_c$ when A_1 is terminated, and Phase 2 wherein A_2 and T playout the *Game of Angle*. The Value function, V_{A_2}, of Phase 2 determines, in part, the equilibrium strategies in Phase 1.

We use the notation $\mathbf{z}_c \equiv \mathbf{z}\left(t_c\right)$ generally. The Hamiltonian is

$$\mathscr{H} = \lambda_\beta u_T + \sum_{i=1}^{2} -\lambda_{R_i} \nu \cos\psi_i + \lambda_{\theta_i}\left(\frac{\nu}{R_i}\sin\psi_i - u_T\right). \tag{8}$$

The equilibrium adjoint dynamics are [11]

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial \mathscr{H}}{\partial \mathbf{z}} = \begin{bmatrix} -\frac{\nu}{R_1^2}\lambda_{\theta_1}\sin\psi_1 \\ 0 \\ -\frac{\nu}{R_1^2}\lambda_{\theta_1}\sin\psi_1 \\ 0 \\ 0 \end{bmatrix}, \tag{9}$$

and thus λ_{θ_1}, λ_{θ_2}, and λ_β are constant. The transversality condition yields the adjoint values at terminal time [11]

$$\begin{aligned} \boldsymbol{\lambda}_c^\top &= \frac{\partial \Phi}{\partial \mathbf{z}_c} + \mu \frac{\partial \phi}{\partial \mathbf{z}_c} \\ &= \begin{bmatrix} 0 & 0 & \frac{\partial V_{A_2}}{\partial R_{2_c}} & \frac{\partial V_{A_2}}{\partial \theta_{2_c}} & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{10}$$

Let the adjoints of A_2's *Game of Angle* be written

$$\boldsymbol{\sigma}^\top \equiv \begin{bmatrix} \sigma_R & \sigma_\theta \end{bmatrix} = \begin{bmatrix} \frac{\partial V_{A_2}}{\partial R_2} & \frac{\partial V_{A_2}}{\partial \theta_2} \end{bmatrix}.$$

Notice that $\lambda_{\beta_c} = 0$ and $\dot{\lambda}_\beta = 0$, thus $\lambda_\beta = 0$ for all $t \in [0, t_c]$. Similarly, $\lambda_{\theta_1} = \mu$ and $\lambda_{\theta_2} = \sigma_\theta$ for all $t \in [0, t_c]$. Substituting the values of λ_β, λ_{θ_1}, and λ_{θ_2}, the Hamiltonian becomes

$$\begin{aligned} \mathscr{H} = &-\lambda_{R_1}\nu\cos\psi_1 + \mu\left(\frac{\nu}{R_1}\sin\psi_1 - u_T\right) \\ &- \lambda_{R_2}\nu\cos\psi_2 + \sigma_\theta\left(\frac{\nu}{R_2}\sin\psi_2 - u_T\right). \end{aligned} \tag{11}$$

The Hamiltonian is a separable function of the controls u_T and ψ_1, ψ_2, and thus *Isaacs' condition* [10], [12] holds:

$$\min_{u_T} \max_{\psi_1, \psi_2} \mathscr{H} = \max_{\psi_1, \psi_2} \min_{u_T} \mathscr{H}.$$

The following result applies generally to differential games based on these dynamics with a well-defined terminal cost functional and terminal surface; it arises mainly as a consequence of the fact that the Attackers have simple motion (i.e., single integrator dynamics). Most of the later results in this paper rely heavily on the following:

Lemma 1 (Equilibrium Controls are Constant)**.** *For any differential game with kinematics described by* (1) *and a Mayer-type cost functional, the equilibrium strategies of all the agents are constant. In particular, each Attacker's equilibrium trajectory is a straight line (in the Cartesian plane), and the Turret's control is either always clockwise or always counterclockwise.*

Proof. Given that the cost functional is of Mayer-type, the Hamiltonian for the system (1) is

$$\mathscr{H} = \lambda_\beta u_T + \sum_{i=1}^{2} \lambda_{x_i}\nu\cos\hat{\psi}_i + \lambda_{y_i}\nu\sin\hat{\psi}_i \tag{12}$$

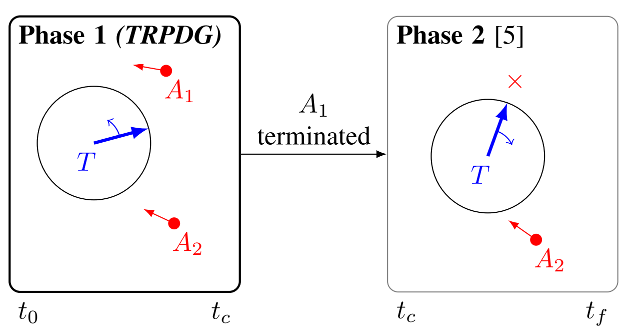


Fig. 2. Abstract depiction of the scenario; in Phase 1 T pursues A_1 while A_2 seeks advantageous position for Phase 2, and Phase 2 is the remaining one-Attacker *Game of Angle*.

Let $\hat{\boldsymbol{\lambda}} \equiv \begin{bmatrix} \lambda_{x_R} & \lambda_{y_R} & \lambda_{x_P} & \lambda_{y_P} & \lambda_\beta \end{bmatrix}^\top$ be the adjoint vector in the Cartesian frame. The equilibrium adjoint dynamics are given by [11]

$$\dot{\hat{\boldsymbol{\lambda}}} = -\frac{\partial \mathscr{H}}{\partial \hat{\mathbf{z}}} = 0. \tag{13}$$

Without loss of generality, suppose that the Attackers seek to maximize the cost functional while the Turret seeks to minimize it. The equilibrium controls are

$$\cos \hat{\psi}_i^* = \frac{\lambda_{x_i}}{\sqrt{\lambda_{x_i}^2 + \lambda_{y_i}^2}}, \ \sin \hat{\psi}_i^* = \frac{\lambda_{y_i}}{\sqrt{\lambda_{x_i}^2 + \lambda_{y_i}^2}}, \ i = 1, 2 \tag{14}$$

$$u_T^* = -\operatorname{sign} \lambda_\beta. \tag{15}$$

Because the equilibrium adjoint dynamics are 0, $\boldsymbol{\lambda}$ is constant, and thus u_T^* and $\hat{\psi}_i^*$ for $i = 1, 2$ are also constant. Since $\hat{\psi}_i$ are defined relative to the positive x-axis, the Attackers' trajectories are straight lines in the Cartesian plane. □

A. Equilibrium Turret & Runner Strategies

Lemma 2 (Equilibrium Turret Strategy)**.** *In the differential game defined by the kinematics,* (2)*, cost functional,* (4)*, and terminal surface,* (5) *the Turret's strategy is*

$$u_T^*(t) = k, \qquad k \in \{-1, 1\}, \forall t \in [0, t_c]. \tag{16}$$

Proof. The fact that k is a constant is due to Lemma 1. The Turret must minimize the Hamiltonian, (11) – in order to do so, we see that

$$u_T^*(t) = \arg\min_{u_T} \mathscr{H} = \operatorname{sign}(\mu + \sigma_\theta).$$

Again, both μ and σ_θ are constant. The sign function ensures that $k \in \{-1, 1\}$. □

Lemma 3 (Equilibrium Runner Strategy)**.** *In the differential game defined by the kinematics,* (2)*, cost functional,* (4)*, and terminal surface,* (5)*, Attacker 1's trajectory is a straight line perpendicular to the Turret's line of sight at the time of termination.*

Proof. Attacker 1 maximizes the Hamiltonian, (11), which occurs when the vector $\begin{bmatrix} \cos \psi_1 & \sin \psi_1 \end{bmatrix}^\top$ is parallel with the vector $\begin{bmatrix} -\lambda_{R_1} & \frac{\mu}{R_1} \end{bmatrix}^\top$. Therefore,

$$\cos \psi_1^* = \frac{-\lambda_{R_1}}{\sqrt{\lambda_{R_1}^2 + \frac{\mu^2}{R_1^2}}}, \qquad \sin \psi_1^* = \frac{\mu}{R_1 \sqrt{\lambda_{R_1}^2 + \frac{\mu^2}{R_1^2}}}. \tag{17}$$

At terminal time, $\lambda_{R_1}(t_c) = 0$ from (10), which implies $\cos \psi_{1_c}^* = 0$. Thus A_1's terminal heading is $\psi_{1_c}^* \in \left\{ \frac{\pi}{2}, -\frac{\pi}{2} \right\}$, and is perpendicular to T's line of sight since $\theta_{1_c} = 0$. The fact that A_1's trajectory is a straight line in the Cartesian coordinate system is due to Lemma 1. □

It remains to show in which direction (either CCW or CW) both the Turret and Runner should travel. In the present case, wherein $A_1, A_2 \notin \mathscr{R}_A$, the biggest benefit for the Attacker team comes when the Runner, A_1, keeps the Turret occupied for as long as possible, thereby giving the Penetrator, A_2, a chance to reach an advantageous position before T starts pursuing A_2 in earnest.

Lemma 4. *The sign of the equilibrium Turret and Runner control inputs are such that*

$$\operatorname{sign}(u_T^*) = \operatorname{sign}(\psi_1^*) = \operatorname{sign}(\theta_1). \tag{18}$$

Proof. In order to limit A_2's ability to improve V_{A_2}, T must terminate A_1 as quickly as possible. To this end, T must minimize $\dot{\theta}_1$ (e.g., when $\theta_1 > 0$), while A_1 maximizes it. Suppose that $u_T^* = -\operatorname{sign}(\theta_1)$. Since $\theta_1 \in [-\pi, \pi]$, by definition, the Turret has a longer distance to cover $(2\pi - \theta_1)$ in this direction, compared to $|\theta_1| < \pi$ to reach A_1's starting position. Thus setting $u_T^* = \operatorname{sign}(\theta_1)$ drives $\theta_1 \to 0$ faster. Suppose that $\operatorname{sign}(\psi_1^*) = -\operatorname{sign}(u_T^*)$. By doing so, A_1 has a component of velocity *towards* the T, and thus could maximize $\dot{\theta}_1$ by setting $\operatorname{sign}(\psi_1^*) = \operatorname{sign}(u_T^*)$. □

B. Equilibrium Penetrator Strategy

The presence of the Dispersal Surface in the one-Attacker game [3], [5] creates an interesting situation in this two-Attacker variant. When the state of a system lies on a Dispersal Surface, the equilibrium controls of one or more agents is non-unique [10]. In the case of the one-Attacker game, when $\theta = \pi$, there is symmetry in the system such that the T could chase A either counterclockwise (CCW) or clockwise (CW) and resulting Value of the *Game of Angle* would be the same [5]. The consequence of the Dispersal Surface is that the one-Attacker Value function V_{A_2} is not smooth along the surface; thus the adjoint vector, $\boldsymbol{\sigma}$, is undefined along the surface. Therefore, A_2's terminal heading, defined by (20) and (10) as $\psi_{2_c}^* = \tan^{-1} -\sigma_\theta / \sigma_R$, is not well-defined either. There are two cases: (1) $\theta_{2_c} \neq \pi$ and $\boldsymbol{\sigma}$ is well-defined (the regular case), and (2) $\theta_{2_c} = \pi$ and $\boldsymbol{\sigma}$ is undefined (the singular case).

Lemma 5 (Regular Equilibrium Penetrator Strategy)**.** *In the differential game defined by the kinematics,* (2)*, cost functional,* (4)*, and terminal surface,* (5) *Attacker 2's equilibrium trajectory is a straight line that is aligned with its Game of Angle equilibrium trajectory at terminal time wherever the Game of Angle adjoints σ_R and σ_θ are defined. Moreover, A_2's control strategy is given by [3], [5]*

$$\psi_2^* = \operatorname{sign}(\theta_{2_c}) \sin^{-1}\left(\frac{\nu}{R_2}\right). \tag{19}$$

Proof. Attacker 2 maximizes the Hamiltonian, (11), which occurs when the vector $\begin{bmatrix} \cos \psi_2 & \sin \psi_2 \end{bmatrix}^\top$ is parallel with the vector $\begin{bmatrix} -\lambda_{R_2} & \sigma_\theta \end{bmatrix}^\top$:

$$\cos \psi_2^* = \frac{-\lambda_{R_2}}{\sqrt{\lambda_{R_2}^2 + \frac{\sigma_\theta^2}{R_2^2}}}, \qquad \sin \psi_2^* = \frac{\sigma_\theta}{R_2 \sqrt{\lambda_{R_2}^2 + \frac{\sigma_\theta^2}{R_2^2}}}. \tag{20}$$

At final time, $\lambda_{R_2} = \sigma_R$ (due to (10)) and thus $\tan \psi_2^* = -\sigma_\theta / \sigma_R$. Thus, at final time, A_2's heading is identical to the equilibrium Attacker heading from the one-Attacker scenario [5]. Furthermore, A_2's trajectory is a straight line in the Cartesian coordinate frame due to Lemma 1, just as it is in the one-Attacker scenario. Therefore, A_2's regular state feedback equilibrium control is given by (19). □

The geometric interpretation of the following Lemma is that the Penetrator's equilibrium trajectory never crosses the $\beta + \pi$ radial. In cases where (19) *would* cause this, the Runner, instead, takes a shallower angle such that $\theta_{2_c} = \pm\pi$.

Lemma 6 (Singular Penetrator Strategy)**.** *In the differential game defined by the kinematics,* (2)*, cost functional,* (4)*, and terminal surface,* (5) *a family of Attacker 2 singular trajectories exist which terminate at* $\theta_{2_c} = \pi$*, with* $R_{2_c} > 1$*. These trajectories are straight lines with the following state feedback strategy*

$$\psi_2^* = \sin^{-1}\left(\frac{\chi\nu}{R_2}\right). \tag{21}$$

where $\chi \in [-1, 1]$ *and* $\text{sign}(\chi) = \text{sign}(\theta_{2_c})$.

Proof. First, recall that the trajectories are straight lines in the Cartesian coordinate frame due to Lemma 1. The general form of the one-Attacker equilibrium control is given in Lemma 9, in the Appendix:

$$\psi_2^* = \sin^{-1}\left(\frac{\text{sign}(\sigma_\theta)\,\nu}{R_2}\right).$$

However, when $\theta_2 = \pi$, the term $\text{sign}(\sigma_\theta)$ is undefined because the Value function V_{A_2} is not differentiable on the Dispersal Surface. We replace the quantity $\text{sign}(\sigma_\theta)$ with a variable χ. When $\chi = \pm 1$, the solution exactly corresponds to the limiting case of the regular equilibrium trajectories described in Lemma 5 where $\sin\psi_2^* = \pm\frac{\nu}{R_2}$. If $|\chi| > 1$, the approach angle to the point $(R_2, \theta_2) = (R_{2_c}, \pi)$ would be steeper. Backwards integrating from (R_{2_c}, π) with an angle $|\psi_2| > \sin^{-1}\left(\frac{\nu}{R_2}\right)$ would push the state of the system into a region that is filled with regular equilibrium trajectories – see Fig. 3. Therefore, it must be the case that $\chi \in [-1, 1]$. The sign of χ is governed by the sign of θ_{2_c} as in the regular trajectory case. □

C. Full solution

Figure 4 shows the state trajectory in the Cartesian coordinate frame for a regular trajectory (with $\theta_{2_c} \neq \pi$) and for a singular trajectory (with $\theta_{2_c} = \pi$). Attacker 1, the Runner, has a trajectory which is perpendicular to the Turret's line of sight at the time of termination. In the regular case, Attacker 2, the Penetrator, has a trajectory which is aimed at the tangent of a circle of radius ν; once A_1 is terminated, A_2 would continue along this course all the way to the target circle. In the singular case, Attacker 2 prefers not to cross the $\theta_2 = \pi$ radial at $t = t_c$ and therefore has taken a shallower angle to end up at $\theta_{2_c} = \pi$. From here, Attacker 2 takes either the upper or lower trajectory depending on T's choice of rotation after terminating A_1 (CW or CCW, respectively).

Although it wasn't explicitly stated in the problem formulation, we require that $\theta_2 \neq 0$ for all $t \in [0, t_c]$ because, otherwise, the Penetrator would have been terminated while T was *en route* to terminate the Runner. The limiting case

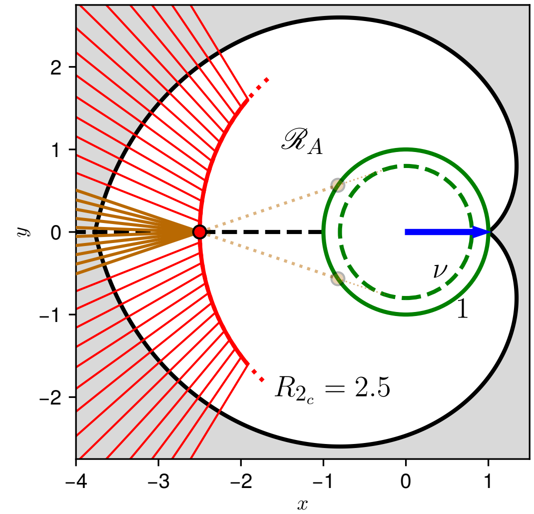
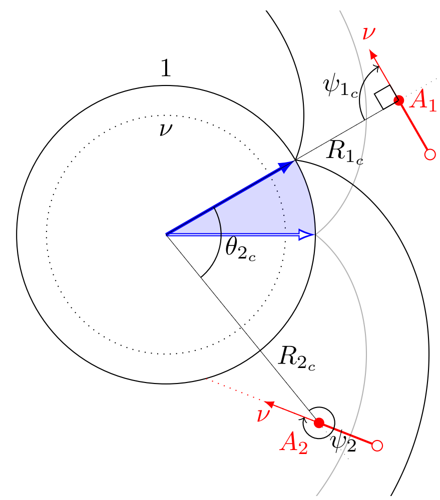
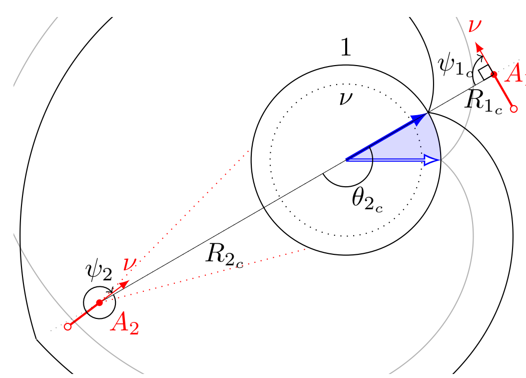


Fig. 3. Attacker 2 regular (red) and singular (dark orange) trajectories. The target circle is green; the dashed inner circle is a circle of radius ν (= 0.8). Note the extension of each regular A_2 trajectory are tangential to the ν circle. The position of the Turret at the time of termination of A_1 is shown by a blue arrow. A family of trajectories is shown wherein $R_{2_c} = 2.5$. Singular A_2 trajectories terminate on the dashed black Dispersal Surface. In the second phase of the scenario, A_2 terminates at either dark orange filled circle depending on T's choice of CCW or CW.



(a) regular



(b) singular

Fig. 4. Representative solutions for the (a) regular and (b) singular cases. Initial Attacker and Turret positions are denoted by open circles and an arrow, respectively; terminal positions are filled. The boundary of $\mathscr{R}_A$ is shown at $t = 0$ (grey) and at $t = t_c$ (black).

occurs when $\text{sign}(\theta_2) = \text{sign}(\theta_1)$ and $\theta_2 \to 0$ precisely at the moment of termination of A_1.

In order for A_2 to win, it must reach $\mathscr{R}_{A_c} \equiv \mathscr{R}_A(t_c)$, i.e., the one-on-one Attacker win region at terminal time. The limiting case occurs when $A_{2_c} \in \partial\mathscr{R}_{A_c}$ where $\partial\mathscr{R}_{A_c}$ is the boundary of the one-on-one Attacker win region at terminal time. That is, the Penetrator is just barely able to satisfy the necessary condition to win (i.e., reach the target) in the second phase of the engagement. Note that $\partial\mathscr{R}_{A_c}$ is the zero-level set of the cost functional, V_{A_2}, and thus the equilibrium Penetrator trajectories terminating at a point on $\partial\mathscr{R}_{A_c}$ are normal to the surface. The other limiting case is when A_2 reaches the target exactly when A_1 is terminated.

Define $\mathscr{R}_{2A}$ as the set of states in which A_2 can be guaranteed to win, i.e., the set of states in which $A_2 \in \mathscr{R}_A$ within t_c time while avoiding premature termination. One boundary of this two-on-one win region, $\partial\mathscr{R}_{2A}$, can be constructed geometrically by setting A_2 on $\partial\mathscr{R}_{A_c}$ and backwards integrating the equilibrium Penetrator strategy ((19) for $\theta_{2_c} \neq \pm\pi$, and (21) for $\theta_{2_c} = \pm\pi$). The other boundary is obtained by setting $R_2 = 1$ and backwards integrating. Care must be taken to eliminate terminal A_2 positions which result in A_2 paths which start and end inside the sector swept by the T's motion (which would result in premature termination.) Figure 5 shows a slice of $\mathscr{R}_{2A}$ for a particular initial Turret position (β) and A_1 position ((R_1, θ_1)).

It's clear from Fig. 5 and Eqs. (19) and (21) that the solution depends on β_c (from which θ_{2_c} is measured), or equivalently, the terminal time, t_c. From Lemmas 2 and 3, along with Lemma 4, we know that A_1 has a component of velocity directed away from T and terminates perpendicular to T's line of sight under optimal play, while T moves in the direction of A_1 at its maximum turn rate. Thus T must cover an angular sector at least $|\theta_1|$. For the Turret, angle traveled and time are equivalent since the Turret's turn rate and the target circle radius are both 1. Let $\gamma \geq 0$ be the amount of additional angle the Turret must cover to terminate the Runner. Then $t_D = |\theta_1| + \gamma$ is time of arrival of the Turret to the candidate terminal position. The Runner's trajectory to the candidate terminal configuration covers an angular sector γ and is perpendicular to T's line of sight in the terminal configuration. See Fig. 6 for a diagram depicting the geometry. Thus $t_A = \frac{1}{\nu} R_1 \sin\gamma$ is the time of arrival of the Runner to the candidate terminal position. In the limiting case, the terminal Runner distance is $R_{1_{\min}} = 1$, which gives an upper bound for γ:

$$\gamma_{\max} = \cos^{-1}\left(\frac{1}{R_1}\right).$$

Now, define the time difference of arrival to the terminal configuration as

$$\begin{aligned} \tau(\gamma) &\equiv t_D(\gamma) - t_A(\gamma) \\ &= |\theta_1| + \gamma - \frac{1}{\nu} R_1 \sin\gamma, \end{aligned} \tag{22}$$

with $\gamma \in [0, \cos^{-1}(\frac{1}{R_1})]$. Clearly it would be suboptimal for the Runner to reach a point, stop, and wait for the Turret to reach that point (i.e., $\tau > 0$); similarly, if the Turret arrives before the Runner (i.e., $\tau < 0$) the Turret would have had to pass the Runner *en route*. Thus, for equilibrium, it must be the case that both agents arrive in the terminal configuration simultaneously, i.e., $\tau^* = 0$.

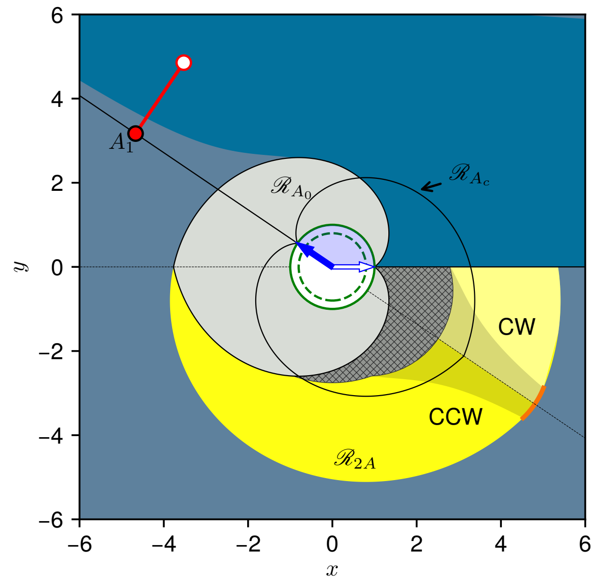


Fig. 5. A partitioning of the state space for particular β, R_1, and θ_1. The T and A_1 trajectories start at the open circles and end at the closed circles. The *Game of Kind* surface θ_{GoK} is drawn at $t = 0$ and at $t = t_c$. Note we do not consider A_2 positions beginning within $\mathscr{R}_{A_0}$, marked by light grey, nor positions in which A_2 wins before t_c, marked by hatched grey. The yellow region represents $\mathscr{R}_{2A}$, the set of A_2 initial conditions which yield a win. In the light shaded portion, A_2's motion has a clockwise component, otherwise it has a counter-clockwise component. The dark shaded portion is filled with singular trajectories which terminate on $\theta_{2_c} = \pm\pi$. There is a segment of $\partial\mathscr{R}_{2A}$ which is a circular arc, marked by orange, which is the locus of extremal A_2 singular initial conditions. Premature termination would occur for any A_2 positions beginning in the bright blue region, and the faded blue region represents positions in which $\mathscr{R}_{A_c}$ cannot be reached; T wins in either case.

Lemma 7. *The function, $\tau(\gamma)$, (22), which represents the time difference of arrival of the Runner and Turret to a candidate terminal configuration, has a unique zero, γ^*, on the interval* $[0, \cos^{-1}(1/R_1)]$.

Proof. First, (22) is a continuous function of γ since γ and $\sin\gamma$ are both continuous. For the lower bound of τ, we have $\tau(0) = |\theta_1|$, and thus $\tau(0) > 0$. In other words the Runner arrives first – in fact, it travels zero distance, whereas the Turret covers $|\theta_1|$ distance. For the upper bound, we will show that $\tau(\gamma_{\max}) < 0$ by contradiction. Suppose that $\tau(\gamma_{\max}) > 0$, that is, the Runner arrives to the candidate terminal configuration before the Turret. The upper bound, $\gamma_{\max}$ is derived from the limiting case where $R_{1_c} \to 1$. This would mean the Runner was able to reach the target circle before the Turret could align with it which contradicts the assumption that $A_1 \notin \mathscr{R}_A$ (which is embedded in the assumption that $\mathbf{z} \in \Omega$). Therefore, from the Intermediate Value Theorem, the function $\tau(\gamma)$ crosses zero on the interval

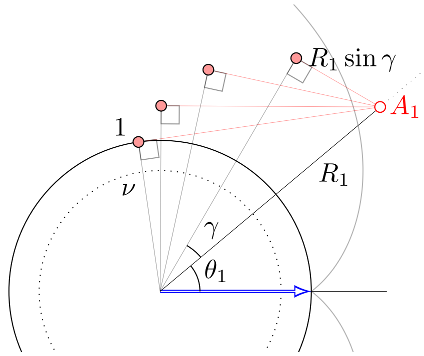


Fig. 6. Relevant geometry for the determination of terminal time t_c. Open circles represent initial positions and the closed red circles indicate candidate terminal configurations for A_1.

$[0, \cos^{-1}(1/R_1)]$.

Also, $\partial\tau/\partial\gamma = 1 - R_1/\nu\cos\gamma$ which is strictly negative on the interval $[0, \cos^{-1}(1/R_1)]$ since $R_1/\nu > 1$ and $\cos\gamma > 0$ on the interval. Thus $\tau(\gamma)$ is monotonic on the interval, which implies that the zero crossing is unique. □

Because of the uniqueness of γ^* in which $\tau(\gamma^*) = 0$ many standard root-finding methods are suitable for computing it. The terminal time is simply

$$t_c = |\theta_1| + \gamma^*. \tag{23}$$

With the value of t_c computed, we obtain $\beta_c = \beta + t_c \,\text{sign}\,(\theta_1)$. From Fig. 5 we see that the effect of sign (θ_{2_c}) in (19) and (21) is that the Runner's motion (at least in $\mathscr{R}_A$) has a component of velocity towards the $\beta_c + \pi$ radial. The interpretation is that the Runner seeks to end up *behind* the Turret at terminal time, which is an advantageous position for the *Game of Angle*. Thus, under equilibrium play by all the agents, the terminal state is

$$\mathbf{z}_c = \begin{bmatrix} R_{1_c} \\ \theta_{1_c} \\ R_{2_c} \\ \theta_{2_c} \\ \beta_c \end{bmatrix} = \begin{bmatrix} R_1 \cos\left(t_c - |\theta_1|\right) \\ 0 \\ \sqrt{\left(\frac{\nu^2 \chi t_c}{R_2}\right)^2 + \left(R_2 - \nu t_c \sqrt{1 - \frac{\chi^2\nu^2}{R_2^2}}\right)^2} \\ \theta_2 - \text{sign}\,(\theta_1)\, t_c + \sin^{-1}\left(\frac{\chi\nu^2 t_c}{R_2 R_{2_c}}\right) \\ \beta + \text{sign}\,(\theta_1)\, t_c \end{bmatrix}, \tag{24}$$

where $\chi \in \{-1, 1\}$ for regular trajectories, $\chi \in [-1, 1]$ for singular trajectories, and

$$\text{sign}\,\chi = \text{sign}\,(\theta_{2_c}) = \text{sign}\,\xi, \tag{25}$$

where $\xi \in [-\pi, \pi]$ is A_2's angle-to-go to the $\beta_c + \pi$ radial,

$$\xi = -\bmod\left(\theta_2 - \text{sign}\,(\theta_1)\, t_c,\ 2\pi\right) + \pi. \tag{26}$$

The trajectory is singular if A_2's regular strategy, (19), would cause it cross the $\beta_c + \pi$ radial, which occurs if

$$\sin^{-1}\left(\frac{\nu^2 t_c}{R_p\sqrt{R_p^2 + \nu^2 t_c^2 - 2R_p\nu t_c\sqrt{1 - \frac{\nu^2}{R_p^2}}}}\right) > |\xi|. \tag{27}$$

Note the LHS of the above expression is the angular sector swept (w.r.t. the origin) by A_2's regular strategy in t_c time. If the trajectory is singular, then, by definition $\theta_{2_c} = \pi$ (or $-\pi$, equivalently). The Law of Sines gives the following relationships:

$$\frac{R_{2_c}}{\sin\psi_2} = \frac{\nu t_c}{\sin\xi} = \frac{R_2}{\sin\left(\pi - |\psi_2| - |\xi|\right)}.$$

The singular A_2 heading is

$$\psi_2 = \text{sign}\,(\xi)\left(\sin^{-1}\left(\frac{R_2\sin|\xi|}{\nu t_c}\right) - |\xi|\right), \tag{28}$$

and the singular terminal A_2 distance is

$$R_{2_c} = \frac{\nu t_c \sin\psi_2}{\sin\xi}. \tag{29}$$

Finally, the Value function is

$$V(\mathbf{z}) = |\theta_{2_c}| - \theta_{GoK}\left(R_{2_c}\right) \tag{30}$$

where (R_{2_c}, θ_{2_c}) is given by (24) and θ_{GoK} is defined in (32).

Attackers 1 and 2 simply aim at their respective terminal point from (24), and the Turret rotates towards Attacker 1. Of course, one or more agents could (to their detriment) deviate from the strategy which would necessitate recomputing the solution in practice. For discrete time systems, for example, it is recommended for the agent implementing its equilibrium strategy to recompute the solution at each time step.

IV. Conclusion

In this paper, we have introduced the two-Attacker, one-Turret circular target guarding problem. Our focus was on a region of the state space in which neither Attacker can guarantee to reach the target, individually. We posed and solved a differential game in the case that one Attacker can guarantee to reach the target after its teammate has been terminated by the Turret. There are two immediate problems to address: 1) the solution over the remaining state space (when A_2 starts in the win region, and when A_2 cannot reach the target at all) and 2) which Attacker the Turret should pursue first (i.e., deciding who is the Runner and who is the Penetrator). Beyond these, the solution of the two-Attacker case will be instrumental in addressing the many-Attacker case.

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Appendix

A. Both Attackers Win

Consider the one-Attacker, one-Turret scenario analyzed in [3], [5]. The region of win for the Attacker, i.e., wherein the Attacker is guaranteed to reach the target circle under optimal play is defined as [5]

$$\mathscr{R}_A \equiv \left\{(R, \theta) \ \middle|\ \theta > \theta_{GoK}(R)\right\}, \tag{31}$$

where

$$\theta_{GoK}(R) = \sqrt{\frac{R^2}{\nu^2} - 1} + \sin^{-1}\left(\frac{\nu}{R}\right) - \sqrt{\frac{1}{\nu^2} - 1} - \sin^{-1}\nu \tag{32}$$

Lemma 8. *In the two-Attacker, one-Turret scenario with kinematics given by* (2)*, both Attackers are guaranteed to reach the target circle under optimal play, that is,* $R_{1_f} = R_{2_f} = 1$ *if and only if* $A_1, A_2 \in \mathscr{R}_A$.

Proof. Optimal play is given by the respective one-Attacker, one-Turret equilibrium control [3], [5]

$$\psi_i^* = \operatorname{sign}(\theta_i)\sin^{-1}\left(\frac{\nu}{R_i}\right).$$

The fact that $A_1, A_2 \in \mathscr{R}_A \implies R_{1_f} = R_{2_f} = 1$ is due to each Attacker being able to win individually; the presence of additional Attackers does not aid the Turret in any way – both A_1 and A_2 are able to win. We now prove that $R_{1_f} = R_{2_f} = 1 \implies A_1, A_2 \in \mathscr{R}_A$. Suppose $A_i \notin \mathscr{R}_A$, the Turret could choose to implement its one-on-one strategy $u_T = \operatorname{sign}(\theta_i)$ against A_i and be guaranteed to terminate A_i with $R_i > 1$. □

B. One or more Attackers Lose

Let the Turret's one-on-one win region be defined $\mathscr{R}_T = \mathscr{R}_A^c$. The trivial case occurs when $A_i \in \mathscr{R}_A$ and $A_j \notin \mathscr{R}_A$ for $i, j \in \{1, 2\}$, $i \neq j$. Clearly, A_i can guarantee a win while T can guarantee termination of A_j. The construction and solution of a *Game of Degree* in this region of the state space is left for future work.

When $A_1, A_2 \notin \mathscr{R}_A$, there is a region of the state space in which one the Attackers can win and a region in which neither can win. The former is analyzed in this paper in detail; the analysis of what the agents should do in the latter region is left for future work.

C. The One-Attacker, One-Turret Differential Game

Lemma 9 (Form of the one-Attacker strategy)**.** *The one-Attacker game, with kinematics* $\dot{\mathbf{z}} = \begin{bmatrix}\dot{R}_2 & \dot{\theta}_2 & \dot{\beta}\end{bmatrix}^\top$*, Value function* $V_{A_2} = \max_{\psi_2}\min_{u_T}|\theta_{2_f}|$*, and terminal surface* $\phi = R_{2_f} - 1 = 0$ *has an equilibrium Attacker 2 strategy of the form*

$$\psi^* = \sin^{-1}\left(\frac{\operatorname{sign}(\sigma_\theta)\,\nu}{R_2}\right). \tag{33}$$

Proof. The Hamiltonian is

$$\mathscr{H} = -\sigma_R \nu \cos\psi_2 + \sigma_\theta\left(\frac{\nu}{R_2}\sin\psi_2 - u_T\right) + \sigma_\beta u_T, \tag{34}$$

and the adjoint dynamics for the θ_2 and β states are

$$\dot{\sigma}_\theta = -\frac{\partial \mathscr{H}}{\partial \theta_2} = 0, \qquad \dot{\sigma}_\beta = -\frac{\partial \mathscr{H}}{\partial \beta} = 0. \tag{35}$$

At final time $t = t_f$, the transversality condition yields the terminal adjoint value for the β state

$$\sigma_{\beta_f} = \frac{\partial \Phi}{\partial \beta_f} + \mu\frac{\partial \phi}{\partial \beta_f} = 0, \tag{36}$$

where $\phi \equiv |\theta_{2_f}|$. Thus $\sigma_\beta = 0$ for all $t \in [0, t_f]$. Attacker 2 wishes to maximize the Hamiltonian, while the Turret seeks to minimize it, giving

$$\sin\psi_2^* = \frac{\sigma_\theta}{R_2\sqrt{\sigma_R^2 + \frac{\sigma_\theta^2}{R_2^2}}}, \qquad u_T^* = \operatorname{sign}(\sigma_\theta). \tag{37}$$

Substituting (36) and (37) into (34) gives

$$\mathscr{H} = \nu\sqrt{\sigma_R^2 + \frac{\sigma_\theta^2}{R_2^2}} - |\sigma_\theta| \tag{38}$$

The terminal Hamiltonian value is

$$\mathscr{H}_f = -\frac{\partial \Phi}{\partial t_f} - \mu\frac{\partial \phi}{\partial t_f} = 0. \tag{39}$$

Since the state dynamics are time-autonomous, $\mathscr{H} = 0$ for all $t \in [0, t_f]$. Substituting into (38) and solving for σ_R^2 gives

$$\sigma_R^2 = \frac{\sigma_\theta^2}{\nu^2} - \frac{\sigma_\theta^2}{R_2^2}. \tag{40}$$

Substituting into (37) yields (33). □

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